# Complex Manifolds

**Recall**: One can easily def a vector field on the plane $R^2$.

例 $\vec{E}(x,y) = (xy^2, xy^2)$ where $(x,y)$: coordinate of $R^2$.

But how can one def a vector field on $S^2$? Martin P.311

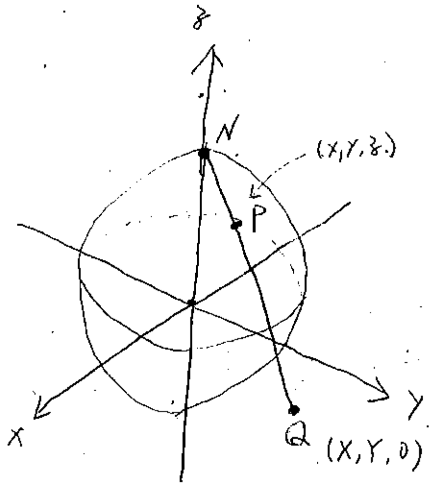
⊛ { A cross-section of the tangent bundle $TS^2$ }

It turns out that one needs to introduce at least two coordinate systems, say 2 stereographic coordinates on $S^2$.

- Stereographic coordinate $(X,Y)$ of $S^2$

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta$$

$$\begin{cases} \theta = \tan^{-1}\left(\frac{(x^2+y^2)^{1/2}}{z}\right), \quad \varphi = \tan^{-1}\frac{y}{x} \\ X = \frac{x}{1-z}, \quad Y = \frac{y}{1-z} \end{cases}$$




$$\Rightarrow \quad X = \cot\frac{\theta}{2}\cos\varphi, \quad Y = \cot\frac{\theta}{2}\sin\varphi$$

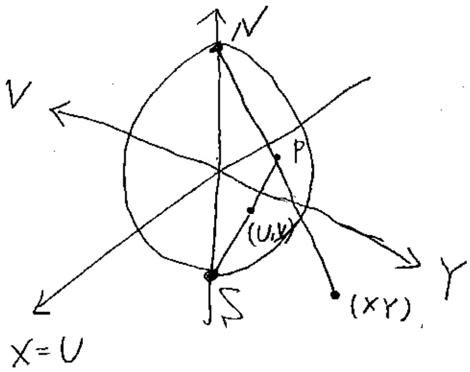
**Note**: $(\theta, \varphi)$ 不好座標：
① $\varphi: 0 \to 2\pi$ 不連續
② $\vec{i}_\varphi\Big|_{\text{poles}} = ?$

$(X,Y)$ 也不好：N点座標 = ??

One way out! 引進 2 組座標 $(X,Y)$ & $(U,V)$

$(X,Y)$ 可以涵蓋 $S^2-\{N\}$

$(U,V)$ ...... $S^2-\{S\}$




那麼 $(X,Y)$ 跟 $(U,V)$ 的關係是？

P点可用 $(X,Y)$ or $(U,V)$ 座標表之。

$\boxed{\text{Ans}}$
$$\begin{cases} (X,Y) = \left(\dfrac{x}{1-z}, \dfrac{y}{1-z}\right) \\ (U,V) = \left(\dfrac{x}{1+z}, \dfrac{-y}{1+z}\right) \end{cases}$$
: Cover $S^2$ with an atlas of two charts

trick：令
$$\begin{cases} Z = X+iY, & \bar{Z} = X-iY \\ W = U+iV, & \bar{W} = U-iV \end{cases}$$

則
$$W = \frac{x-iy}{1+z} = \frac{1-z}{1+z}(X-iY) = \frac{X-iY}{X^2+Y^2} = \frac{1}{Z}$$

$$\therefore \quad U(X,Y) = \frac{X}{X^2+Y^2}$$
$$V(X,Y) = \frac{-Y}{X^2+Y^2}$$

$U(X,Y)$ 與 $V(X,Y)$ 在赤道附近均為 $C^\infty$ 函數。

smoothly!

$\therefore$ $S^2$ 稱 Real (Diff) Manifold

Now we can def a vector field on $S^2$

by using two charts $(X, Y)$ and $(U, V)$.

例 驗證 $X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}$, $-U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V}$

定义了一個 vector field on $S^2$.

proof: We need to prove that the two expressions agree on the intersection of the two coordinate systems.

$$-U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V} = -\frac{X}{X^2+Y^2}\left(\frac{\partial X}{\partial U}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial U}\frac{\partial}{\partial Y}\right)$$

$$-\left(\frac{-Y}{X^2+Y^2}\right)\left(\frac{\partial X}{\partial V}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial V}\frac{\partial}{\partial Y}\right).$$

$\boxed{\frac{\partial}{\partial X}}$ : $-\frac{X}{X^2+Y^2}\frac{V^2-U^2}{[U^2+V^2]^2} + \frac{Y}{X^2+Y^2}\frac{-2UV}{[U^2+V^2]^2}$

$$= -X(X^2+Y^2)(V^2-U^2) - Y(X^2+Y^2)2UV$$

$$= X^2+Y^2\left[-X\frac{Y^2-X^2}{[X^2+Y^2]^2} + Y\frac{2XY}{[X^2+Y^2]^2}\right]$$

$$= \frac{XY^2+X^3}{X^2+Y^2} = X.$$

$\boxed{\frac{\partial}{\partial Y}}$ :

$$-\frac{X}{X^2+Y^2}\frac{2UV}{[U^2+V^2]^2}+\frac{Y}{X^2+Y^2}\frac{V^2-U^2}{[U^2+V^2]^2}$$

$$= -X(X^2+Y^2)2UV + Y(X^2+Y^2)(V^2-U^2)$$

$$= X^2+Y^2\left[-X\frac{(-2XY)}{[X^2+Y^2]^2}+Y\cdot\frac{Y^2-X^2}{[X^2+Y^2]^2}\right]$$

$$= \frac{YX^2+Y^3}{X^2+Y^2} = Y$$

∴ 在赤道附近

$$-U\frac{\partial}{\partial U}-V\frac{\partial}{\partial V} = X\frac{\partial}{\partial X}+Y\frac{\partial}{\partial Y}$$ #.

$\boxed{\text{Note}}$ There are 2 zeros of the vector field.

North pole $(U, V) = (0, 0)$

south pole $(X, Y) = (0, 0)$ ("hairy ball" theorem)

**Def** A Real (Diff) Manifold $M$

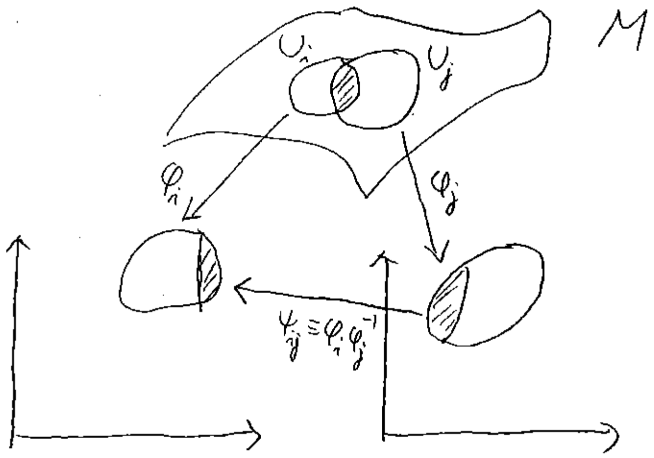
见黄 P.184 "广义曲面"

(1) A family of pairs $\{(U_i, \varphi_i)\}$

$\bigcup_i U_i = M$, $\varphi_i : U_i \to R^m$ 为 $U_i$ 的 smoothly 局部座標

(2) for $U_i \cap U_j \neq \phi$, the map $\psi_{ij} \equiv \varphi_i \circ \varphi_j^{-1} \in C^\infty$

$$\varphi_i \circ \varphi_j^{-1} : \quad \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$$




the transition function (coordinate transformation) is smooth.

**例** $S^2$: $\{(U_1, (X, Y)), (U_2, (U, V))\}$ ← an atlas

2 charts (or patches)

$\boxed{\text{Def}}$ $\{$A complex Manifold $M\}$

(1) ...... $\varphi_i : U_i \to C^m$ (複座標)

(2) ........ $\psi_{ij} \equiv \varphi_i \cdot \varphi_j^{-1} \in$ holomorphic (much more stronger, rather than $C^\infty$

$\boxed{\text{例}}$ $S^2$ is a complex manifold !

$$Z = X + iY \qquad W = U + iV$$

transition function $W = \dfrac{1}{Z}$ analytic on $U_1 \cap U_2$ #

$\therefore$ $S^2$ is a complex manifold $\sim$ Riemann sphere $C \cup \{\infty\}$ #

$\boxed{\text{例}}$ $CP^1$ : space of complex lines through the origin in $C^2$.

or $(z^1, z^2) = C^2 - (0,0)$ ( $z_i$ are not all zero)

and identify $(z^1, z^2) \approx \lambda (z^1, z^2)$ for any non-zero complex $\lambda$ #

(homogeneous coordinates)

2 coordinates : $U_1 = \{(z^1, z^2) \mid z^1 \neq 0\}$ $\quad U_2 = \{(z^1, z^2) \mid z^2 \neq 0\}$

$$\xi_1 = \frac{z^2}{z^1} \qquad\qquad \xi_2 = \frac{z^1}{z^2}$$

(inhomogeneous coordinates)

on the overlap : $\xi_1 = \dfrac{1}{\xi_2}$ analytic on $U_1 \cap U_2$

$\therefore$ $CP^1 = S^2$, the Riemann sphere #

A complex manifold

$\boxed{例}$ $CP^2$ : $(z^1, z^2, z^3) \approx \lambda(z^1, z^2, z^3)$

3 - patches $U_1 = \{(z^1, z^2, z^3) \mid z^1 \neq 0\}$ $U_2 = \{(z^1, z^2, z^3) \mid z^2 \neq 0\}$ $U_3 = \{(z^1, z^2, z^3) \mid z^3 \neq 0\}$

coordinates

homo

$$\boxed{\xi_j^m \equiv \frac{z^m}{z^j}}$$

inhomo

$$\begin{cases} \xi_1^2 = \dfrac{z^2}{z^1} \\ \xi_1^3 = \dfrac{z^3}{z^1} \end{cases} \qquad \begin{cases} \xi_2^1 = \dfrac{z^1}{z^2} \\ \xi_2^3 = \dfrac{z^3}{z^2} \end{cases} \qquad \begin{cases} \xi_3^1 = \dfrac{z^1}{z^3} \\ \xi_3^2 = \dfrac{z^2}{z^3} \end{cases}$$

$(\xi_1^1 \equiv 1)$ $(\xi_2^2 \equiv 1)$ $(\xi_3^3 \equiv 1)$

transition functions

$U_1 \cap U_2$ : $(z^1 \neq 0$ 且 $z^2 \neq 0)$
$$\begin{cases} \xi_1^2 = \dfrac{1}{\xi_2^1} \\ \xi_1^3 = \dfrac{\xi_2^3}{\xi_2^1} \end{cases},$$

$U_1 \cap U_3$ : $(z^1 \neq 0$ 且 $z^3 \neq 0)$
$$\begin{cases} \xi_1^2 = \dfrac{\xi_3^2}{\xi_3^1} \\ \xi_1^3 = \dfrac{1}{\xi_3^1} \end{cases}$$

$U_2 \cap U_3$ : $(z^2 \neq 0$ 且 $z^3 \neq 0)$
$$\begin{cases} \xi_2^1 = \dfrac{\xi_3^1}{\xi_3^2} \\ \xi_2^3 = \dfrac{1}{\xi_3^2} \end{cases},$$

That is, on the overlap $U_j \cap U_k$, we have

$$\boxed{\xi_j^m = \frac{\xi_k^m}{\xi_k^j}} \quad \text{analytic on } U_j \cap U_k$$

(transition functions)

$\boxed{例}$ The above can be easily generalized to $CP^n$ which is a complex manifold.

$$C^n \longrightarrow CP^n \qquad CP^n = C^n \cup CP^{n-1}$$

緊緻化

例 One can def Real projective space $RP^n$

$(x^1, x^2, \cdots x^{n+1}) \approx \lambda (x^1, x^2, \cdots x^{n+1})$ for any non-zero $\lambda \in R$.

$RP^n$ is a real Diff manifold

$$R^n \xrightarrow[\text{紧致化}]{} RP^n \qquad RP^n = R^n \cup RP^{n-1} = S^n/Z_2$$

例 $RP^1 = S^1$, $RP^2 = S^2/Z_2$, $RP^3 = S^3/Z_2 = SO(3)$

- A <u>holomorphic vector field</u> of the <u>holomorphic tangent bundle</u>

over $CP^1$

P.3 中的 vector field on $S^2$ 可寫成一 holomorphic

function on $CP^1$

$$Z = X + iY.$$

$$\therefore \quad X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} = X\frac{\partial}{\partial Z}\frac{\partial Z}{\partial X} + Y\frac{\partial}{\partial Z}\frac{\partial Z}{\partial Y}$$

$$= X\frac{\partial}{\partial Z} + iY\frac{\partial}{\partial Z} = Z\frac{\partial}{\partial Z}.$$

同理 $\quad -U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V} = -W\frac{\partial}{\partial W} \qquad (W = U + iV)$

transition function $\quad W = \frac{1}{Z}$ analytic on $U_1 \cap U_2$

- Complex manifold vs. Real manifold

→ Existence of a globally defined "almost complex structure"

Let $T_pM$ the tangent space of a complex manifold with $\dim_{\mathbb{C}} M = m$.

$T_pM$ is spanned by $2m$ vectors

( $m, n, \cdots$ real coordinates )

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^m} ; \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \cdots, \frac{\partial}{\partial y^m} \right\} \qquad \triangle$$

where $z^\mu = x^\mu + i y^\mu$ are the coordinates of $p$ in a chart $(U, \varphi)$.

We can instead def $2m$ basis vectors of $T_pM$ $(1 \le \mu \le m)$

$$\frac{\partial}{\partial z^\mu} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - i \frac{\partial}{\partial y^\mu}\right) \qquad \frac{\partial}{\partial \bar{z}^\mu} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} + i \frac{\partial}{\partial y^\mu}\right). \quad \circledast$$

( $\mu, \nu, \cdots$ complex coordinates )

Note that $\overline{\frac{\partial}{\partial z^\mu}} = \frac{\partial}{\partial \bar{z}^\mu}$.

One can def a linear map $J_p : T_pM \to T_pM$

$$J_p \cdot \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu}, \quad J_p \frac{\partial}{\partial y^\mu} = - \frac{\partial}{\partial x^\mu}. \quad \cdots \text{①}$$

Note that $J_p$ is a real tensor of type $(1,1)$

and $J_p^2 = -I_p \leftarrow$ (Identity map on $T_pM$)

$$J_p = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad \text{w.r.t. basis } \triangle$$

For a complex manifold, all components of $J_p$ are constant at any point, and $J$ is a globally well defined tensor field.
(called the almost complex structure of a complex manifold)

Indeed, let $(U, \varphi)$, $(V, \psi)$ two overlapping Charts

$$\varphi(p) = z^\mu = x^\mu + i y^\mu \qquad \psi(p) = w^\mu = u^\mu + i v^\mu$$

on $U \cap V$, $z^\mu = z^\mu(w)$ satisfy Cauchy-Riemann relations (C.R.)

$$\Rightarrow \quad J_p \frac{\partial}{\partial u^\mu} = J_p \left( \frac{\partial x^\nu}{\partial u^\mu} \frac{\partial}{\partial x^\nu} + \frac{\partial y^\nu}{\partial u^\mu} \frac{\partial}{\partial y^\nu} \right)$$

$$\downarrow \text{C.R} \qquad \downarrow \text{C.R}$$

$$= \frac{\partial y^\nu}{\partial v^\mu} \frac{\partial}{\partial y^\nu} + (-) \frac{\partial x^\nu}{\partial v^\mu} (-) \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial v^\mu}$$

Similarly $\quad J_p \frac{\partial}{\partial v^\mu} = - \frac{\partial}{\partial u^\mu}$

same with ① in p.10.

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Note that

$$J_p \frac{\partial}{\partial z^\mu} = i \frac{\partial}{\partial z^\mu}, \quad J_p \frac{\partial}{\partial \bar{z}^\mu} = -i \frac{\partial}{\partial \bar{z}^\mu}$$

and $J_p \doteq \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}$ w.r.t basis ⊛ in p.10

$\boxed{\text{Def}}$ If a real manifold $M$ admits a globally defined $J_p$ with $J_p^2 = -I_p$, then $M$ is called an almost complex manifold

$\boxed{\text{Real manifold} \supset \text{Almost complex manifold} \supset \text{Complex manifold}}$

例 $S^4$ is a Real manifold but NOT an Almost complex manifold.

(Steenrod 1951)

例 $S^6$ is an Almost complex manifold but NOT a Complex manifold

(Fröhlicher 1955)

# Hermitian manifolds

Similar to: Real manifold $\xrightarrow[\text{with } g_{mn}]{\text{equipped}}$ Riemannian manifold, (if $\Gamma^{\lambda}_{[mn]}=0$)

we want to give complex manifold a metric.

In general, the components of $g$ w.r.t. the basis ⊛ can have the following non-zero elements

$$\begin{cases} g_{\mu\nu}(p) = g_p\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) \\ g_{\mu\bar{\nu}}(p) = g_p\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) \\ g_{\bar{\mu}\nu}(p) = g_p\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}\right) \\ g_{\bar{\mu}\bar{\nu}}(p) = g_p\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) \end{cases}, \quad \text{and} \quad \begin{cases} g_{\mu\nu} = g_{\nu\mu},\ g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu},\ g_{\bar{\mu}\nu} = g_{\nu\bar{\mu}} \\ \overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu},\ \overline{g_{\mu\nu}} = g_{\bar{\mu}\bar{\nu}} \end{cases}$$

Hermitian metric: a Riemannian metric $g$ of a Complex manifold $M$

若 $\boxed{g_p(J_pX, J_pY) = g_p(X, Y) \quad \text{or} \quad g_{mn} = J_m{}^k J_n{}^l g_{kl}}$

at each point $p \in M$ and for any $X, Y \in T_pM$, 則稱

$g$ a Hermitian metric.

Hermitian manifold: $(M, g)$ is called a Hermitian manifold.

$$\boxed{\text{Hermiticity is a restriction on the metric and NOT on the manifold}}$$

Indeed, **Theorem** A complex manifold always admits a Hermitian metric.

proof: Let $g$ be a Riemannian metric of a complex manifold $M$.

Def a new metric $\hat{g}_p(X,Y) \equiv \frac{1}{2}[g_p(X,Y) + g_p(JX,JY)]$

$$\Rightarrow \begin{cases} \hat{g}_p(JX,JY) = \hat{g}(X,Y) \\ \hat{g}_p \text{ is positive definite if } g_p \text{ is.} \end{cases} \Rightarrow \hat{g}_p \text{ is a Hermitian metric}$$

✱

⊛ Let $g$ be a Hermitian metric of a complex manifold $M$

$$g_{\mu\nu} = g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = g\left(J\frac{\partial}{\partial z^\mu}, J\frac{\partial}{\partial z^\nu}\right) = (i)^2 g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = -g_{\mu\nu}$$

同理 $g_{\bar\mu\bar\nu} = -g_{\bar\mu\bar\nu} \Rightarrow \underline{g_{\mu\nu} = 0 = g_{\bar\mu\bar\nu}}$

$\therefore$ Hermitian metric: $g_{mn} = \begin{pmatrix} 0 & g_{\mu\bar\nu} \\ g_{\bar\mu\nu} & 0 \end{pmatrix}$

⊛ $$\boxed{g = g_{\mu\bar\nu}\, dz^\mu \otimes d\bar{z}^\nu + g_{\bar\mu\nu}\, d\bar{z}^\mu \otimes dz^\nu}$$

cf P.16
Kähler form

"Hermitian": $\overline{g_{\mu\bar\nu}} = (g_{\bar\mu\nu})^*, \quad \overline{g_{\bar\mu\nu}} = (g_{\mu\bar\nu})^*$

$\boxed{\text{kähler form}}$ On a Hermitian manifold, the Almost complex structure $J_m{}^n$ defines a natural two-form :

(A) 由 $g_{mn} = J_m{}^k J_n{}^l g_{kl}$

(Candelas) 兩邊乘 $J_r{}^m$ , $\boxed{\text{定义 } J_{mn} \equiv J_m{}^k g_{kn}}$ (Candelas 用 J 表之)

$$\Rightarrow \quad J_{rn} = -\delta_r{}^k J_n{}^l g_{kl} = -J_n{}^l g_{rl} = -J_n{}^l g_{lr}$$
$$= -J_{nr}.$$

$\boxed{J_{mn} = -J_{nm}}$ 稱 kähler form of a Hermitian metric $g$.

(B)

(Nakahara) Define a tensor field $\Omega$

$$\boxed{\text{定义 } \Omega_p(X,Y) = g_p(J_pX, Y)}$$

$$\Rightarrow \quad \Omega(X,Y) = g(JX,Y) = g(J^2X, JY) = -g(JY,X) = -\Omega(Y,X)$$

$\Omega$ is antisymmetric $\Rightarrow$ two-form , kähler form

$\Omega$ is a two-form of bidegree $(1,1)$.

Indeed,

(B)
$$\begin{cases} \Omega\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = g\left(J\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = i g_{\mu\nu} = 0 \\ \Omega\left(\frac{\partial}{\partial \bar z^\mu}, \frac{\partial}{\partial \bar z^\nu}\right) = g\left(J\frac{\partial}{\partial \bar z^\mu}, \frac{\partial}{\partial \bar z^\nu}\right) = -i g_{\bar\mu\bar\nu} = 0 \\ \Omega\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar z^\nu}\right) = i g_{\mu\bar\nu} = -\Omega\left(\frac{\partial}{\partial \bar z^\nu}, \frac{\partial}{\partial z^\mu}\right) \end{cases}$$

$$\therefore \quad \Omega = i g_{\mu\bar\nu}\, dz^\mu \otimes d\bar z^\nu - i g_{\bar\nu\mu}\, d\bar z^\nu \otimes dz^\mu = i g_{\mu\bar\nu}\, dz^\mu \wedge d\bar z^\nu$$
#

(A)

$$J_{\mu\nu} = J_\mu{}^m g_{m\nu} = 0 \quad \longleftarrow \quad J = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}$$

$$J_{\bar\mu\bar\nu} = J_{\bar\mu}{}^m g_{m\bar\nu} = 0$$

$$J_{\mu\bar\nu} = J_\mu{}^m g_{m\bar\nu} = i g_{\mu\bar\nu}, \quad J_{\bar\mu\nu} = J_{\bar\mu}{}^m g_{m\nu} = -i g_{\bar\mu\nu}$$

$$\therefore \quad J = J_{\mu\bar\nu}\, dz^\mu \otimes d\bar z^\nu + J_{\bar\nu\mu}\, d\bar z^\nu \otimes dz^\mu$$
$$= i g_{\mu\bar\nu}\, dz^\mu \otimes d\bar z^\nu - i g_{\bar\nu\mu}\, d\bar z^\nu \otimes dz^\mu$$

$$\Rightarrow \circledast \boxed{J = i g_{\mu\bar\nu}\, dz^\mu \wedge d\bar z^\nu}$$
#

Kähler form

比 P.14 Hermition metric

## • an Application of kähler form

<u>Theorem</u> : Any Hermitian manifold, and hence any complex manifold, is <u>orientable</u>.

proof : Let $g$ be a <u>Hermitian metric</u> of $M$, $\dim_{\mathbb{C}} M = m$

(Nakahara)

We can always choose $\{\hat{e}_1, J\hat{e}_1, \hat{e}_2, J\hat{e}_2, \cdots, \hat{e}_m, J\hat{e}_m\}$ to be an orthonormal basis.

Indeed, if $g(\hat{e}_1,\hat{e}_1)=1 \Rightarrow$

$$g(J\hat{e}_1, J\hat{e}_1) = g(\hat{e}_1,\hat{e}_1) = 1$$

$$g(J\hat{e}_1, \hat{e}_1) = -g(\hat{e}_1, J\hat{e}_1) = -g(J\hat{e}_1, \hat{e}_1)$$

$$\therefore\ g(J\hat{e}_1, \hat{e}_1) = 0$$

同理 $g(\hat{e}_1, J\hat{e}_1) = 0$.

Choose $\hat{e}_2 \perp \begin{cases} \hat{e}_1 \\ J\hat{e}_1 \end{cases}$ $\cdots$ $\hat{e}_2, J\hat{e}_2$, 正交 $\cdots$

Now consider the <u>kahler form $\Omega$</u> of the <u>Hermitian metric $g$</u>, and construct the $2m$-form

$$\Omega \wedge \Omega \wedge \cdots \wedge \Omega .$$

One can show the it is a nowhere vanishing $2m$-form, and can be served as $\boxed{\text{a volume element}}$

<u>Indeed</u> $\Omega(\hat{e}_i, J\hat{e}_j) = g(J\hat{e}_i, J\hat{e}_j) = \delta_{ij}$ , $\Omega(\hat{e}_i, \hat{e}_j) = 0 = \Omega(J\hat{e}_i, J\hat{e}_j)$

$$\Rightarrow \underbrace{\Omega \wedge \cdots \wedge \Omega}_{m\text{个}} (\hat{e}_1, J\hat{e}_1, \cdots, \hat{e}_m, J\hat{e}_m) \overset{\because}{=} \sum_P \Omega(\hat{e}_{P(1)}, J\hat{e}_{P(1)}) \cdots \Omega(\hat{e}_{P(m)}, J\hat{e}_{P(m)})$$

$$= m!\ \Omega(\hat{e}_1, J\hat{e}_1) \cdots \Omega(\hat{e}_m, J\hat{e}_m) = m!$$

#

proof : (Candelas) p.45

$$\underbrace{J \wedge J \wedge \cdots \wedge J}_{m\text{ 個}} = i^m g_{\mu_1\bar{\nu}_1} g_{\mu_2\bar{\nu}_2} \cdots g_{\mu_m\bar{\nu}_m} dz^{\mu_1} \wedge dz^{\bar{\nu}_1} \wedge \cdots \wedge dz^{\mu_m} \wedge dz^{\bar{\nu}_m}$$

$\boxed{\text{J cannot be exact}}$

$$= i^m \epsilon^{\mu_1\mu_2\cdots\mu_m} \overbrace{g_{\mu_1\bar{\nu}_1} \cdots g_{\mu_m\bar{\nu}_m}}^{\epsilon^{\bar{\nu}_1\bar{\nu}_2\cdots\bar{\nu}_m}} dz^1 dz^{\bar{1}} \cdots dz^m dz^{\bar{m}}$$

(or kähler manifold) $\therefore\ b^{2m} \geq 1$ indeed, $b^{2p} \geq 1$ $(1 \leq p \leq m)$ Nakahara p.296

$$= i^m m!\, \det(g_{\mu\bar{\nu}})\, dz^1 dz^{\bar{1}} \cdots dz^m dz^{\bar{m}}$$

Now $g$ is a Hermitian metric $\Rightarrow$

$$g_{\ell n} = \begin{pmatrix} 0 & g_{\mu\bar{\nu}} \\ g_{\bar{\mu}\nu} & 0 \end{pmatrix}, \quad \overline{g_{\bar{\mu}\nu}} = g_{\mu\nu}$$

$$\therefore\ \det g_{\ell n} = g = (\det g_{\mu\bar{\nu}})^2 \Rightarrow \det g_{\mu\bar{\nu}} = \sqrt{g}$$

$\therefore$ $J \wedge J \wedge \cdots \wedge J$ is proportional to the volume form 証

## Hermitian connection

Require metric compatibility & $\Gamma$ (mixed indices) $= 0$

$$\downarrow$$

$$\nabla_m g_{nr} = \partial_m g_{nr} - \Gamma_{mn}{}^{k} g_{kr} - \Gamma_{mr}{}^{k} g_{nk} = 0$$

取 $(m,n,r) = (\mu, \iota, \bar{\rho})$

$$\Rightarrow \quad \partial_\mu g_{\iota\bar{\rho}} - \Gamma_{\mu\iota}{}^{\alpha} g_{\alpha\bar{\rho}} = 0$$

取 $(m,n,r) = (\bar{\mu}, \iota, \bar{\rho})$

$$\Rightarrow \quad \partial_{\bar{\mu}} g_{\iota\bar{\rho}} - \Gamma_{\bar{\mu}\bar{\rho}}{}^{\bar{\alpha}} g_{\iota\bar{\alpha}} = 0$$

(此时 Real manifold 比较: metric compatibility & torsion fr

(可解出 $\Gamma$)

$\therefore$ Hermitian connection is $\boxed{\text{uniquely}}$ fixed for a given Hermitian metric to be

$$\boxed{\Gamma_{\mu\iota}{}^{\alpha} = g^{\bar{\lambda}\alpha} \partial_\mu g_{\iota\bar{\lambda}}, \quad \Gamma_{\bar{\mu}\bar{\iota}}{}^{\bar{\alpha}} = g^{\bar{\alpha}\lambda} \partial_{\bar{\mu}} g_{\lambda\bar{\iota}}.}$$

Hermitian manifold

<u>Theorem</u> Ⓐ : $\nabla_m J_{nr} = 0$ (or $\nabla_m J_n{}^r = 0$) w.r.t. Hermitian connection leads to the same relations!

almost complex structure.

- Riemann tensor & Ricci-form

Let $g$ be a Hermitian metric and $\Gamma$ the corresponding Hermitian connection, then the only nonzero components of $R_{mnkl}$ are those that are mixed in both the first and last pairs of indices

$$\boxed{R_{\mu\bar{\nu}\rho\bar{\sigma}} \, , \; R_{\bar{\mu}\nu\rho\bar{\sigma}} \, , \; R_{\mu\bar{\nu}\bar{\rho}\sigma} \, , \; R_{\bar{\mu}\nu\bar{\rho}\sigma}} \quad \cdots \text{ Hermitian manifold}$$

or $\left( R^{\bar{\mu}}{}_{\bar{\nu}\rho\bar{\sigma}} \, , \; R^{\mu}{}_{\nu\rho\bar{\sigma}} \, , \; R^{\bar{\mu}}{}_{\bar{\nu}\bar{\rho}\sigma} \, , \; R^{\mu}{}_{\nu\bar{\rho}\sigma} \right)$

Note : $\because \quad R^{\bar{\mu}}{}_{\bar{\nu}\bar{\sigma}\rho} = - R^{\bar{\mu}}{}_{\bar{\nu}\rho\bar{\sigma}} \, , \quad R^{\mu}{}_{\nu\rho\bar{\sigma}} = - R^{\mu}{}_{\nu\bar{\sigma}\rho}$

$\therefore$ The only indep components are

$$R^{\mu}{}_{\nu\bar{\rho}\sigma} \quad \text{and} \quad R^{\bar{\mu}}{}_{\bar{\nu}\rho\bar{\sigma}} = - \overline{R^{\mu}{}_{\nu\bar{\rho}\sigma}}$$

with
$$\begin{cases} R^{\mu}{}_{\nu\bar{\rho}\sigma} = \partial_{\bar{\rho}} \Gamma^{\mu}_{\nu\sigma} = \partial_{\bar{\rho}} \left( g^{\bar{\lambda}\mu} \partial_{\nu} g_{\sigma\bar{\lambda}} \right) \\ R^{\bar{\mu}}{}_{\bar{\nu}\rho\bar{\sigma}} = \partial_{\rho} \Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\sigma}} = \partial_{\rho} \left( g^{\bar{\mu}\lambda} \partial_{\bar{\nu}} g_{\lambda\bar{\sigma}} \right) \end{cases}$$

## • The Ricci form

On a Hermitian manifold, the Kähler form is associated with the Hermitian metric

| metric | Kähler form |
|---|---|
| $g = g_{\mu\bar{\nu}}\, dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\mu}\nu}\, d\bar{z}^\mu \otimes dz^\nu$ | $J = i\, g_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu$. |

Now the form associated with Riemann tensor is the Ricci form

| Riemann tensor | Ricci form (NOT to be confused with Ricci tensor !!) |
|---|---|
| $R_{\mu\bar{\nu}\rho\bar{\sigma}}$ etc. | (Nakahara) |

• $$\mathcal{R}_{\mu\bar{\nu}} \equiv R^{k}{}_{k\mu\bar{\nu}} = -\partial_{\bar{\nu}}(g^{k\bar{\lambda}}\partial_\mu g_{k\bar{\lambda}})$$

$$= -\partial_{\bar{\nu}}\partial_\mu \ln G \quad (\text{locally !!})$$

where $G \equiv \det g_{\mu\bar{\nu}} = \sqrt{g}$.

• $$\mathcal{R} \equiv i\,\mathcal{R}_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu = i\,\partial\bar{\partial}\ln G.$$

(Candelas)

• $$\mathcal{R} = \frac{1}{4} R_{mnkl} J^{kl}\, dx^m \wedge dx^n$$

$$= i\, R_{\mu\bar{\nu}\bar{\rho}}{}^{\bar{\rho}}\, dz^\mu \wedge d\bar{z}^\nu$$

$$= i\,\partial\bar{\partial}\ln G$$

**Note**

① $\delta G = G\, g^{\mu\nu}\, \delta g_{\mu\nu}$ under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

proof: 由 matrix identity $\ln(\det g_{\mu\nu}) = \mathrm{tr}(\ln g_{\mu\nu})$ (对角化 $g_{\mu\nu}$ 後可得)

$$\Rightarrow \quad \delta G \cdot G^{-1} = g^{\mu\nu}\, \delta g_{\mu\nu}$$

$$\Rightarrow \quad \delta G = G\, g^{\mu\nu}\, \delta g_{\mu\nu}$$

$$\therefore \quad g^{k\bar\lambda}\, \partial_\mu g_{k\bar\lambda} = \partial_\mu \ln G$$

② $\mathcal{R}$ is a real form.

Indeed, $\overline{\mathcal{R}} = -i\,\overline{\partial\bar\partial} \ln G = -i\,\bar\partial\partial \ln G = i\,\partial\bar\partial \ln G = \mathcal{R}$.

③ <u>The Dolbeault operators</u> $\partial$, $\bar\partial$

$$d = \partial + \bar\partial \quad , \quad \partial^2 = 0 = \bar\partial^2$$

$$\because \quad d^2 = 0 = (\partial + \bar\partial)^2 \quad \Rightarrow \quad \partial\bar\partial + \bar\partial\partial = 0 .$$

A 2-form $\Omega = \Omega_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^{\bar\beta}$ is $d$-closed

$\Leftrightarrow \quad \Omega = \partial\bar\partial f$ , $f$ a real scalar function.
(locally!)

④ $\mathcal{R}$ is closed, $d\mathcal{R} = 0$.

Indeed, $\because \quad \partial\bar\partial = -\frac{1}{2} d(\partial - \bar\partial)$

$\therefore \quad d\mathcal{R} \propto d^2(\partial - \bar\partial)\ln G = 0$. However $\mathcal{R}$ is NOT exact!

$\mathcal{R}$ defines a non-trivial element

$$C_1(M) \equiv \left[\frac{\mathcal{R}}{2\pi}\right] \in H^2(M, R)$$ called the first Chern Class.

Theorem : The first Chern Class $C_1(M)$ is invariant under a smooth change $g \to g + \delta g$.

proof $\because \delta \ln G = g^{\mu\bar{\nu}} \delta g_{\mu\bar{\nu}}$ ,

$$\therefore \delta \mathcal{R} = i \partial\bar{\partial} g^{\mu\bar{\nu}} \delta g_{\mu\bar{\nu}} = -\tfrac{1}{2}(\partial - \bar{\partial}) i g^{\mu\bar{\nu}} \delta g_{\mu\bar{\nu}} .$$

Now $\because$ $g^{\mu\bar{\nu}} \delta g_{\mu\bar{\nu}}$ is a globally defined scalar,

$\therefore$ $-\frac{1}{2}(\partial - \bar{\partial}) i g^{\mu\bar{\nu}} \delta g_{\mu\bar{\nu}}$ is a well defined 1-form on $M$,

$$\Rightarrow \quad [\mathcal{R}] = [\mathcal{R} + \delta\mathcal{R}] . \quad \#$$

- Kähler manifolds & Kähler diff Geometry.

$\boxed{\text{Def}}$ (*) A Kähler manifold is a Hermitian manifold $(M, g)$ whose Kähler form $\Omega$ is closed : $d\Omega = 0$.

$g$ is called the Kähler metric of $M$.

(NOT all complex manifolds admit Kähler metrics.)

$\boxed{\text{Theorem}}$ (B) A Hermitian manifold $(M, g)$ is a Kähler manifold

$\Leftrightarrow$ $\nabla_M J = 0$ where $\nabla_M$ is the (*) Levi-Civita connection associated with $g$.

$\boxed{\text{Note}}$ Theorems (A) (P.19) & (B)

$\Rightarrow$ In the Kähler manifold, the Riemann structure (**) is compatible with the Hermitian structure.

Indeed, it can be shown that Kähler metric is torsion free. (see P.25)

- Kähler diff Geometry

Let $g$ be a Kähler metric

$$d\Omega = 0 \Rightarrow (\partial + \bar\partial)\, i g_{\mu\bar\iota}\, dz^\mu \wedge d\bar z^\iota$$

$$= i\partial_\lambda g_{\mu\bar\iota}\, dz^\lambda \wedge dz^\mu \wedge d\bar z^\iota + i\partial_{\bar\lambda} g_{\mu\bar\iota}\, d\bar z^\lambda \wedge dz^\mu \wedge d\bar z^\iota$$

$$= \frac{1}{2} i \left(\partial_\lambda g_{\mu\bar\iota} - \partial_\mu g_{\lambda\bar\iota}\right) dz^\lambda \wedge dz^\mu \wedge d\bar z^\iota$$

$$+ \frac{1}{2} i \left(\partial_{\bar\lambda} g_{\mu\bar\iota} - \partial_{\bar\iota} g_{\mu\bar\lambda}\right) d\bar z^\lambda \wedge dz^\mu \wedge d\bar z^\iota = 0,$$

$$\Rightarrow \boxed{\partial_\lambda g_{\mu\bar\iota} = \partial_\mu g_{\lambda\bar\iota}, \quad \partial_{\bar\lambda} g_{\mu\bar\iota} = \partial_{\bar\iota} g_{\mu\bar\lambda}} \text{ -- } \triangle$$

(*) This $_\triangle$ ensures that the Kähler metric is <u>torsion free</u>

$$\begin{cases} T^\lambda{}_{\mu\iota} = g^{\bar\alpha\lambda}\left(\partial_\mu g_{\iota\bar\alpha} - \partial_\iota g_{\mu\bar\alpha}\right) = 0 \\ T^{\bar\lambda}{}_{\bar\mu\bar\iota} = g^{\bar\lambda\alpha}\left(\partial_{\bar\mu} g_{\bar\iota\alpha} - \partial_{\bar\iota} g_{\bar\mu\alpha}\right) = 0. \end{cases}$$

\#

(*) This $_\triangle$ implies that the Riemann tensor has an extra symmetry

$$R^k{}_{\lambda\mu\bar\iota} = -\partial_{\bar\iota}\left(g^{\bar\alpha k}\partial_\mu g_{\lambda\bar\alpha}\right) = -\partial_{\bar\iota}\left(g^{\bar\alpha k}\partial_\lambda g_{\mu\bar\alpha}\right) = R^k{}_{\mu\lambda\bar\iota}.$$

Also $\quad R^{\bar k}{}_{\bar\lambda\bar\mu\iota} = R^{\bar k}{}_{\bar\mu\bar\lambda\iota}, \quad R^k{}_{\lambda\bar\mu\iota} = R^k{}_{\iota\bar\mu\lambda}, \quad R^{\bar k}{}_{\bar\lambda\mu\bar\iota} = R^{\bar k}{}_{\bar\iota\mu\bar\lambda}.$ \#

(*) This implies that the components of the Ricci form agree with $\mathrm{Ric}_{\mu\bar{\iota}}$ (Ricci tensor).

Indeed, $\mathcal{R}_{\mu\bar{\iota}} \equiv R^{k}{}_{k\mu\bar{\iota}} = R^{k}{}_{\mu k\bar{\iota}} = \mathrm{Ric}_{\mu\bar{\iota}}$. #

(物理)
(真空 Einstein eq 的解!!)

$\boxed{\text{Def}}$ : If $\mathrm{Ric} = \mathcal{R} = 0$, the Kähler metric is said to be Ricci-flat.

$\boxed{\text{Theorem}}$ : If $M$ admits a Ricci-flat metric
(Calabi)

$$\Rightarrow \quad c_1(M) = 0 .$$

$\boxed{\text{Theorem}}$ : $\boxed{\text{Calabi - Yau manifold}}$

(conjectured by Calabi (1957)
proved by Yau (1977))

Calabi conjectured that $c_1(M) \overset{\neq 0}{}$ is the only topological obstruction for $M$ to be Ricci-flat,

that is, $c_1(M) = 0 \Rightarrow$ Kähler manifold admits a Ricci-flat metric. *

(uniqueness proved by
Calabi)

(proved by Yau!!!)

## The kähler potential

Let $g$ be a kähler metric of a kahler manifold.

Since $\partial_\lambda g_{\mu\bar{\nu}} = \partial_\mu g_{\lambda\bar{\nu}}$ , $\partial_{\bar{\lambda}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\lambda}}$ ,

it can be shown that (locally)

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K_i \quad \text{on a chart } U_i .$$

$K_i$ is called the kähler potential of a kähler metric.

And $J = i \partial \bar{\partial} K_i$ on a chart $U_i$

Now. $K_i$ cannot be globally well defined.

proof : $\because$ $\partial\bar{\partial} = -\frac{1}{2} d(\partial - \bar{\partial})$

$\therefore$ $J = -\frac{1}{2} d[(\partial - \bar{\partial})K_i]$

if $K_i$ is globally defined $\Rightarrow$ $(\partial - \bar{\partial})K_i$ is globally defined

$\Rightarrow$ $J$ is exact.

But $J$ cannot be exact on a kahler manifold! (see P.18)

for a kahler manifold, $b^{2m} \geq 1$ (or $b^{2m} \neq 0$)

indeed $b^{2p} \geq 1$ , $1 \leq p \leq m$ . (Nakahara P.296)

⊛ Witten P.437

On $U_i \cap U_j$

$$K_j = K_i + f_{ij}(z) + g_{ij}(\bar{z}) \quad .$$

$\boxed{131}$ Any (orientable) complex manifold $M$ with $\dim_{\mathbb{C}} M = 1$

is kahler.

$\because$ a 3-form $d\Omega \equiv 0$ on $M$.

Note : 1-dim compact orientable complex manifolds are known as Riemann surfaces.

$\boxed{131}$

$$\mathbb{C}^m \qquad\qquad \mathbb{R}^{2m}$$

$$\{(z^1, z^2, \cdots z^m)\} \qquad \{(x^1, x^2, \cdots x^m ; y^1, y^2, \cdots, y^m)\}$$

Let $\delta$ be the Euclidean metric of $\mathbb{R}^{2m}$,

$$\begin{cases} \delta\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \delta_{\mu\nu} = \delta\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right), \\ \delta\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu}\right) = 0 \quad . \end{cases}$$

In complex coordinates $(z^\mu = x^\mu + i y^\mu)$

$$\begin{cases} \delta\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = 0 = \delta\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) \quad , \\ \delta\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) = \frac{1}{2}\delta_{\mu\nu} = \delta\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}\right) . \end{cases}$$

Note that $J\dfrac{\partial}{\partial x^\mu} = \dfrac{\partial}{\partial y^\mu}$ , $J\dfrac{\partial}{\partial y^\mu} = -\dfrac{\partial}{\partial x^\mu}$

$\Rightarrow$ $\delta$ is a Hermitian metric

The kahler form is

$$\Omega = \frac{i}{2}\sum_{\mu=1}^{m} dz^\mu \wedge d\bar{z}^\mu = \sum_{\mu=1}^{m} dx^\mu \wedge dy^\mu$$

Clearly, $d\Omega = 0$.

$\therefore$ The Euclidean metric $\delta$ of $R^{2m}$ is a kahler metric of $C^m$. The kahler potential is

$$K = \frac{1}{2}\sum_{\mu=1}^{m} z^\mu \bar{z}^\mu = \frac{1}{2}\sum_{\mu=1}^{m} |z^\mu|^2$$

$\ast$
$\ast\ast$

$\boxed{(2n)}$ $CP^n$ is a kahler manifold.

The coordinates for $U_j$

inhomo $\rightarrow$ $\boxed{\xi_j^m = \dfrac{z^m}{z^j}}$ $\leftarrow$ homo

The transition functions on $U_j \cap U_k$

$$\boxed{\xi_j^m = \frac{\xi_k^m}{\xi_k^j}}$$

Set $$K_j = \log\left(\sum_{m=1}^{n+1} |\xi_j^m|^2\right) \quad \cdots\cdots \quad (\Delta)$$

$$K_j = \log\left(\sum_{m=1}^{n+1} \frac{|\xi_k^m|^2}{|\xi_k^j|^2}\right) = K_k - \log \xi_k^j - \log \bar{\xi}_k^{\bar{j}},$$

$$\therefore \quad \partial\bar{\partial} K_j = \partial\bar{\partial} K_k .$$

On $CP^n$ we choose the Fubini-Study metric

$$g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K_j ,$$

which is globally well defined. The induced Kahler form

$$J = i\,\partial\bar{\partial} K_j .$$

Computations of $J$ & $g_{n\bar{n}}$

By $(\Delta)$, taking $j = n+1$ and writing $\xi_{n+1}^m$ as $\xi^m$

$$\Rightarrow \quad K_{n+1} = \log\left(1 + \sum_{m=1}^{n} |\xi^m|^2\right)$$

$$\therefore \quad \bar{\partial} K_{n+1} = \left(1 + \sum_{m=1}^{n} |\xi^m|^2\right)^{-1} \left(\sum_{m=1}^{n} \xi^m d\bar{\xi}^{\bar{m}}\right)$$

$$\partial\bar{\partial} K_{n+1} = \left(1 + \sum |\xi^m|^2\right)^{-1} \sum d\xi^m \wedge d\bar{\xi}^{\bar{m}}$$
$$- \left(1 + \sum |\xi^m|^2\right)^{-2} \sum \bar{\xi}^n d\xi^n \wedge \sum \xi^m d\bar{\xi}^{\bar{m}}$$

$$\therefore \quad J = \frac{i}{(1+\sum|\xi^m|^2)^2}\left[\delta_{nm}(1+\sum|\xi^m|^2) - \bar{\xi}^n\xi^m\right] d\xi^n \wedge d\bar{\xi}^{\bar{m}}.$$

$$\Rightarrow \quad g_{n\bar{m}} = \frac{\delta_{nm}(1+\sum|\xi^m|^2) - \bar{\xi}^n\xi^m}{(1+\sum|\xi^m|^2)^2}$$

( Fubini-Study metric )

① $g_{n\bar{m}}$ is Hermitian, $\overline{g_{n\bar{m}}} = g_{m\bar{n}}$.

② $g_{n\bar{m}}$ is positive definite.

$$\because \quad \delta_{nm}(1+\sum|\xi^m|^2) v^n\bar{v}^{\bar{m}} - \bar{\xi}^n\xi^m v^n\bar{v}^{\bar{m}}$$

$$= \langle v, v\rangle (1+\langle \xi,\xi\rangle) - |\langle \xi, v\rangle|^2 > 0,$$

by Schwarz's inequality.

$\boxed{例}$ Hypersurfaces in $CP^n$ (GSW 書. P.428, P.436)

Let $P(z^1, z^2, \cdots, z^{n+1})$ be a homogeneous polynomial of degree $k$

$$P(\lambda z^1, \lambda z^2, \cdots, \lambda z^{n+1}) = \lambda^k P(z^1, z^2, \cdots, z^{n+1})$$

$\Rightarrow$ $P = 0$ makes sense in $CP^n$, and defines a complex submanifold of $CP^n$ of dimension $n-1$ called

a degree $k$ hypersurface.

Now, a complex submanifold of a kahler manifold is always a kahler manifold

$\Rightarrow$ $\boxed{\text{Hypersurface in } CP^n \text{ is a kahler manifold !!}}$

例 GSW P. 429

$$P(x, y, z) = x^m + y^m - z^m = 0 \quad \text{in } CP^2$$

defines a compact complex manifold of dimension one

$\Rightarrow$ Riemann Surface of genus $\dfrac{(m-1)(m-2)}{2}$.