Hodge theory for kähler Manifolds

We want to use cohomology groups to study topological properties of a complex manifold. Let's begin with the

Complex differential forms

Let M be a complex manifold with the almost complex structure J_p. In the holomorphic bases

$$J_p = i\, dz^\mu \otimes \frac{\partial}{\partial z^\mu} - i\, d\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu}$$

or

$$J_p = \begin{pmatrix} i\mathbb{I} & 0 \\ 0 & -i\mathbb{I} \end{pmatrix}.$$

It kahler form (see P.16 of the previous note)

$$\Omega = i g_{\mu\bar{\nu}}\, dz^\mu \otimes d\bar{z}^\nu - i g_{\bar{\nu}\mu}\, d\bar{z}^\nu \otimes dz^\mu$$

$$= i g_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu$$

a real 2-form.

(見 Nakahara P.283)

"a (1,1)-form" 見 P.2.

- Def : $\Omega^q_p(M)^{\mathbb{C}}$: the vector space of complex q-forms at p.

  - Let ω, η be two real q-forms on M

two complex q-form $\begin{cases} \xi = \omega + i\eta \\ \lambda = \varphi + i\psi \end{cases}$

Def: $\xi \wedge \lambda = (\omega + i\eta) \wedge (\varphi + i\psi)$

$= (\omega \wedge \varphi - \eta \wedge \psi) + i(\omega \wedge \psi + \eta \wedge \varphi)$

$$d\xi = d\omega + i\,d\eta$$
$$\overline{d\xi} = d\omega - i\,d\eta = d\bar{\xi}$$

- Note that in a complex manifold

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-$$

where $\quad T_pM^{\pm} = \{ Z \in T_pM^{\mathbb{C}} \mid J_pZ = \pm iZ \}$.

$\boxed{\text{Def}}$: (r,s)-form on M, $\dim_{\mathbb{C}} M = m$

Let $\omega \in \Omega^q_p(M)^{\mathbb{C}}$ $(q \le 2m)$, $q = r + s$. non-negative ~~positive~~ integers

Let $V_i \in T_pM^{\mathbb{C}}$ $(1 \le i \le q)$ be vectors in either T_pM^+ or T_pM^-.

$\boxed{\omega \text{ is a } (r,s) \text{ form}}$ $\overset{\text{Def}}{\Longleftrightarrow}$ $\omega(V_1, V_2, \cdots V_q) = 0$ unless $\begin{cases} r \text{ of the } V_i \text{ are in } T_pM^+ \\ s \text{ of the } V_i \text{ are in } T_pM^- \end{cases}$

$\in \Omega^{r,s}_p(M)$

- Since $\langle dz^{\mu}, \frac{\partial}{\partial \bar{z}^{\nu}} \rangle = 0 \Rightarrow dz^{\mu}$ is of bidegree $(1,0)$

  $\langle d\bar{z}^{\mu}, \frac{\partial}{\partial z^{\nu}} \rangle = 0 \Rightarrow d\bar{z}^{\mu}$ ------ $(0,1)$

$\Rightarrow$ The set $\{ dz^{\mu_1} \wedge dz^{\mu_2} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge d\bar{z}^{\nu_2} \wedge \cdots \wedge d\bar{z}^{\nu_s} \}$ is a basis of $\Omega_p^{r,s}(M)$

$$\therefore \quad \omega = \frac{1}{r!s!} \omega_{\mu_1 \cdots \mu_r, \bar{\nu}_1 \cdots \bar{\nu}_s} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_s}.$$

(*) The components of a (r,s) form are totally antisymmetric in the μ and ν ~~separately~~.

- Proposition (1) $\omega \in \Omega^{r,s}(M) \Rightarrow \bar{\omega} \in \Omega^{s,r}(M)$

  (2) $\omega \in \Omega^{r,s}(M)$, $\xi \in \Omega^{r',s'}(M) \Rightarrow \omega \wedge \xi \in \Omega^{r+r', s+s'}(M)$.

(3) $\omega \in \Omega^q(M)^{\mathbb{C}}$ can be uniquely written as

$$\boxed{\omega = \sum_{r+s=q} \omega^{(r,s)}}, \quad \omega^{(r,s)} \in \Omega^{r,s}(M).$$

$$\Rightarrow \quad \Omega^q(M)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{r,s}(M)$$

(4) $$\dim_{\mathbb{R}} \Omega_p^{r,s}(M) = \begin{cases} \binom{m}{r}\binom{m}{s} & 0 \le r,s \le m, \\ 0 & \text{otherwise}. \end{cases} \qquad (\dim_{\mathbb{C}} M = m)$$

Exercise (5) Let z^{μ}, w^{μ} be two overlapping coordinates

則 (r,s) form in z^{μ} coordinate $\Rightarrow$ (r,s) form in w^{μ} coordinate

(6) $\dim_{\mathbb{R}} \Omega_p^q(M)^{\mathbb{C}} = \sum_{r+s=q} \dim_{\mathbb{R}} \Omega_p^{r,s}(M) = \binom{2m}{q}$

- <u>The Dolbeault Operators</u>

Let $\omega \in \Omega^{r,s}(M)$

$$\Rightarrow \quad d\omega = \frac{1}{r!s!}\left(\frac{\partial}{\partial z^\lambda}\omega_{\mu_1\cdots\mu_r\bar{\nu}_1\cdots\bar{\nu}_s}dz^\lambda + \frac{\partial}{\partial\bar{z}^\lambda}\omega_{\mu_1\cdots\mu_r\bar{\nu}_1\cdots\bar{\nu}_s}d\bar{z}^\lambda\right)$$

$$\times\, dz^{\mu_1}\wedge\cdots\wedge dz^{\mu_r}\wedge d\bar{z}^{\nu_1}\wedge\cdots\wedge d\bar{z}^{\nu_s}$$

a mixture of a $(r+1,s)$-form and a $(r,s+1)$-form.

$$\partial : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M)$$
$$\bar{\partial} : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M)$$

are called the <u>Dolbeault operators</u>.

- Let $\omega \in \Omega^q(M)^c$, then.

$$\partial\omega = \sum_{r+s=q}\partial\omega^{(r,s)}, \quad \bar{\partial}\omega = \sum_{r+s=q}\bar{\partial}\omega^{(r,s)}.$$

<u>Theorem</u> : M : complex manifold ; $\omega, \xi \in \Omega^q(M)^c$, then

(1) $\partial\partial\omega = (\partial\bar{\partial}+\bar{\partial}\partial)\omega = \bar{\partial}\bar{\partial}\omega = 0$

(2) $\partial\bar{\omega} = \overline{\bar{\partial}\omega}, \quad \bar{\partial}\bar{\omega} = \overline{\partial\omega}$

(3) $\partial(\omega\wedge\xi) = \partial\omega\wedge\xi + (-)^q\omega\wedge\partial\xi$

$\quad\ \bar{\partial}(\omega\wedge\xi) = \bar{\partial}\omega\wedge\xi + (-)^q\omega\wedge\bar{\partial}\xi$

Def: M : a complex manifold

if $\omega \in \Omega^{r,0}(M)$, $\bar\partial \omega = 0$ $\rightarrow$ ω is called a holomorphic r-form.

例 $r=0$, $\dfrac{\partial f}{\partial \bar z^\lambda} = 0$, for $1 \le \lambda \le m = \dim_{\mathbb{C}} M$.

$\therefore$ f is just a holomorphic function.

例 $\omega \in \Omega^{r,0}(M)$, $\bar\partial \omega = 0$

$$\Rightarrow \frac{\partial}{\partial \bar z^\lambda} \omega_{\mu_1 \cdots \mu_r} = 0 \Rightarrow \omega_{\mu_1 \cdots \mu_r} \text{ are holomorphic functions.}$$

Def: $\dim_{\mathbb{C}} M = m$. the sequence of maps.

$$\Omega^{r,0}(M) \xrightarrow{\bar\partial} \Omega^{r,1}(M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{r,m-1}(M) \xrightarrow{\bar\partial} \Omega^{r,m}(M)$$

is called Dolbeault Complex.

Def: $Z^{r,s}_{\bar\partial}(M)$ = the (r,s)-cocycle = the set of $\bar\partial$-closed (r,s)-forms

$$= \{ \omega \mid \omega \in \Omega^{r,s}(M),\ \bar\partial \omega = 0 \}.$$

$B^{r,s}_{\bar\partial}(M)$ = the (r,s)-coboundary = the set of $\bar\partial$-exact (r,s)-forms

$$= \{ \omega \mid \omega \in \Omega^{r,s}(M) \text{ such that } \omega = \bar\partial \eta \text{ for some } \eta \in \Omega^{r,s-1}(M) \}$$

$\boxed{\text{Def}}$: $H^{r,s}_{\bar\partial}(M) \equiv Z^{r,s}_{\bar\partial}(M) / B^{r,s}_{\bar\partial}(M)$

is called the (r,s)th $\bar\partial$-cohomology group.

(*) The $\bar\partial$-cohomology groups measure the topological non-triviality of a complex manifold M.

- The adjoint operators $\partial^\dagger$ and $\bar\partial^\dagger$

Let M be a Hermitian manifold with $\dim_{\mathbb{C}} M = m$,

$\alpha, \beta \in \Omega^{r,s}(M)$ $(0 \le r, s \le m)$.

Note that $*\beta : \Omega^{r,s} \to \Omega^{m-s,\, m-r}$ since

$$* dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar z^{\nu_1} \wedge \cdots \wedge d\bar z^{\nu_s}$$

$$\sim \epsilon^{\mu_1 \cdots \mu_r}{}_{\bar\mu_{r+1} \cdots \bar\mu_m} \, \epsilon^{\bar\nu_1 \cdots \bar\nu_s}{}_{\nu_{s+1} \cdots \nu_m} \times d\bar z^{\mu_{r+1}} \wedge \cdots d\bar z^{\mu_m} \wedge dz^{\nu_{s+1}} \wedge \cdots \wedge dz^{\nu_m}$$

(the only non-vanishing components in a Hermitian manifold).

$\therefore$ Hodge star $\bar *$: $\Omega^{r,s} \to \Omega^{m-r,\, m-s}$.

$$(\bar * \beta \equiv \overline{* \beta})$$

Def inner product

$$\boxed{(\alpha, \beta) = \int_M \alpha \wedge \bar * \beta}.$$

adjoint operators $\partial^\dagger$, $\bar\partial^\dagger$.

$$(\alpha, \partial\beta) = (\partial^\dagger \alpha, \beta), \quad (\alpha, \bar\partial\beta) = (\bar\partial^\dagger \alpha, \beta).$$

For a complex manifold (an even dim real manifold)

$$\longrightarrow \quad d^\dagger = - * d *$$

proposition
- $\partial^\dagger = -*\bar{\partial}*$, $\bar{\partial}^\dagger = -*\partial*$
- $(\partial^\dagger)^2 = 0 = (\bar{\partial}^\dagger)^2$

- Laplacians and the Hodge theorem.

Similar to p. 9 in the previous lecture ;

Def : A (r,s)-form ω

① is called $\bar{\partial}$-harmonic if $\Delta_{\bar{\partial}}\omega = 0$,

② is called $\bar{\partial}$-closed if $\bar{\partial}\omega = 0$,

③ is called $\bar{\partial}$-co-closed if $\bar{\partial}^\dagger\omega = 0$.

(similar for $\bar{\partial} \to \partial$)

(Def

$$\Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^\dagger)^2 = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$$

$$\Delta_\partial = (\partial + \partial^\dagger)^2 = \partial\partial^\dagger + \partial^\dagger\partial$$

)

on a Hermitian manifold

Theorem : A (r,s)-form is $\bar{\partial}$-harmonic

$\Longleftrightarrow$ ω is $\bar{\partial}$-closed and $\bar{\partial}$-co-closed.

proof :

$$(\omega, \Delta_{\bar{\partial}}\omega) = (\omega, (\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial})\omega)$$

$$= (\bar{\partial}\omega, \bar{\partial}\omega) + (\bar{\partial}^\dagger\omega, \bar{\partial}^\dagger\omega) \geq 0$$

#

$\boxed{\text{Hodge's theorem}}$ $\quad M$: a Hermitian manifold.

$\Omega^{r,s}(M)$ has a unique orthogonal decomposition

$$\Omega^{r,s}(M) = \bar\partial\Omega^{r,s-1}(M) \oplus \bar\partial^{+}\Omega^{r,s+1}(M) \oplus \underset{\substack{\| \\ \{\omega \in \Omega^{r,s}(M) / \Delta_{\bar\partial}\omega = 0\}}}{\mathrm{Harm}^{r,s}_{\bar\partial}(M)}$$

namely an (r,s)-form ω is uniquely expressed as

$$\boxed{\omega = \bar\partial\alpha + \bar\partial^{+}\beta + \gamma}$$

where $\alpha \in \Omega^{r,s-1}(M)$, $\beta \in \Omega^{r,s+1}(M)$ and $\gamma \in \mathrm{Harm}^{r,s}_{\bar\partial}(M)$.

Def : The complex dimension of $H^{r,s}_{\bar\partial}(M)$ is called the

$\boxed{\text{Hodge number } b^{r,s} \text{ (or } h^{r,s}\text{)}}$.

• Laplacians and Hodge numbers of kähler manifold

In general Hermitian manifolds, there exist no relationships among Δ, Δ_{∂} and $\Delta_{\bar\partial}$.

No direct relationships among b^{p} and $b^{r,s}$.

However, for the kähler manifolds (the Levi-Civita connection is compatible with the Hermitian connection !!), we have

Theorem $\quad \Delta = 2\Delta_{\partial} = 2\Delta_{\bar\partial}$ ...... $\circledast$

Application: w is a holomorphic form of degree p

(Def: a diff form of type $(p,0)$ satisfies the condition $\bar\partial w = 0$, or

$w = w_{\mu_1\cdots\mu_p}\, dz^{\mu_1}\wedge\cdots\wedge dz^{\mu_p}$ where $w_{\mu_1\cdots\mu_p}$ are holomorphic functions)

$\Longleftrightarrow$ w is a harmonic form of bidegree $(p,0)$!!

proof: "$\Rightarrow$" $\quad \bar\partial w = 0$, $\bar\partial^{\dagger} w = 0 \Rightarrow \Delta_{\bar\partial} w = 0 \overset{\text{由}\circledast}{\Rightarrow} \Delta w = 0$

"$\Leftarrow$" $\quad \Delta w = 0 \overset{\text{由}\circledast}{\Rightarrow} \Delta_{\bar\partial} w = 0 \overset{\text{由P.8定理}}{\Rightarrow} \bar\partial w = 0$ #

• Hodge numbers of Kähler manifolds

Hodge diamond

$$
\begin{array}{ccccccc}
 & & & b_{00} & & & \\
 & & b_{10} & & b_{01} & & \\
 & b_{20} & & b_{11} & & b_{02} & \\
b_{30} & & b_{21} & & b_{12} & & b_{03} \\
 & b_{31} & & b_{22} & & b_{13} & \\
 & & b_{32} & & b_{23} & & \\
 & & & b_{33} & & &
\end{array}
$$

$(m+1)^2$ Hodge numbers

$m=3$

Theorem Let M be a Kähler manifold with $\dim_{\mathbb{C}} M = m$

then ① $b^{r,s} = b^{s,r}$ ① ⇒ 垂直轴对称

② $b^{r,s} = b^{m-r,\,m-s}$. {①② ⇒ 水平轴对称 or 中心点对称.

proof ① 由 theorem ⊛ in page 10.

② Poincaré duality ! $b^r = b^{n-r}$ ※

• Hodge numbers and Betti numbers

Theorem M : Kähler manifold, $\dim_{\mathbb{C}} M = m$ and $\partial M = \emptyset$

then b^p $(1 \le p \le 2m)$ satisfy the following.

ⓐ $b^p = \sum_{r+s=p} b^{r,s}$

ⓑ b^{2p-1} is even $(1 \le p \le m)$

ⓒ $b^{2p} \ge 1$ $(1 \le p \le m)$.

proof : ⓐ The complexification of $H^p(M)$ is

$$H^p(M)^{\mathbb{C}} = \{ w \mid \Delta w = 0 \}$$

$\because$ M is kähler , by theorem (*) in page 10

$$H^p(M)^{\mathbb{C}} = \bigoplus_{r+s=p} H^{r,s}(M)$$

ⓑ 由 ⓐ and $b^{r,s} = b^{s,r}$

$$b^{2p-1} = \sum_{r+s=2p-1} b^{r,s} = 2 \sum_{\substack{r+s=2p-1 \\ r>s}} b^{r,s}$$

例 in. p. 11 $\quad b_{30} \; b_{21} \; b_{12} \; b_{03} = 2 \, (b_{30} + b_{21})$.

ⓒ Let Ω be the kähler form , $d\Omega = 0$.

Then the volumn form

$$\Omega^p = \underbrace{\Omega \wedge \cdots \wedge \Omega}_{p \text{ 个}}$$

is closed and NOT exact (volumn form).

So ther is at least one non-trivial element of $H^{2p}(M)$

$$\Rightarrow \quad b^{2p} \geq 1 .$$

- Hodge numbers of Calabi-Yau manifolds

例)

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & b_{11} & & 0 & \\ 1 & & b_{21} & & b_{12} & & 1 \\ & 0 & & b_{11} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$m=3$

b_{11}, b_{21} ($b_{11} \geq 1$)

↑ kähler form.

例) <u>K3 manifold</u> $m=2$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

例) Riemann surfaces of genus g.

$m=1$

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

$\therefore$ $b_{00} = 1 = b_{11}$

$b_{12} = g = b_{21}$

$\Rightarrow$ $b_0 = 1$, $b_1 = 2g$, $b_2 = 1$

$\Rightarrow$ $\chi = 1 - 2g + 1 = 2 - 2g$.