Chapter 3

Instanton Solutions in Non-Abelian Gauge Theory

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3.1 Conventions

Gauge fields:

$$
\begin{aligned}
A_\mu &= g A_\mu^a T^a \qquad (3.1)\\
T^a &= \text{anti-hermitian generator}\\
\left[T^a, T^b\right] &= f^{abc} T^c\\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = g F_{\mu\nu}^a T^a \qquad (3.2)\\
F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \qquad (3.3)
\end{aligned}
$$

Normalization: We normalize the generator as

$$\mathrm{Tr} T^a T^b = -\frac{1}{2}\delta^{ab} \qquad (3.4)$$

In the case of $G = SU(2)$, this corresponds precisely to the choice

$$T^a = t^a \equiv \frac{\tau^a}{2i} \qquad (3.5)$$

Covariant derivative:

$$
\begin{aligned}
D_\mu &\equiv \partial_\mu + A_\mu \qquad (3.6)\\
F_{\mu\nu} &= [D_\mu, D_\nu] \qquad (3.7)
\end{aligned}
$$

Inner product notation: Sometimes we use the following inner product notation

$$(T^a, T^b) \equiv \delta^{ab} \quad (= -2\mathrm{Tr} T^a T^b) \qquad (3.8)$$

Euclidean action:

$$
\begin{aligned}
S_E &= \frac{1}{4}\int d^4x\, F_{\mu\nu}^a F_{\mu\nu}^a = -\frac{1}{2g^2}\int d^4x\, \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})\\
&= \frac{1}{4g^2}\int d^4x (F_{\mu\nu}, F_{\mu\nu}) \geq 0 \qquad (3.9)
\end{aligned}
$$

Indices and ϵ-tensor: To conform to some of the important literatures, we use the convention

$$\begin{aligned} \mu, \nu &= 0, 1, 2, 3 \\ \epsilon_{0123} &= 1 \end{aligned} \tag{3.10}$$

Remark: If one uses the convention $\mu = 1 \sim 4$ and $\epsilon_{1234} = 1$, self-dual and anti-self-dual solutions are switched.

3.2 Decomposition $SO(4) = SU(2) \times SU(2)$ and Quaternions

In the following, the instanton for $G = SU(2)$ will play a fundamental role. This solution intertwines the gauge group and the spacetime symmetry group $SO(4)$, which can be decomposed as $SU(2) \times SU(2)$. This decomposition is intimately related to the **quaternion**, which will play a basic role in the ADHM construction.

3.2.1 Decomposition of $SO(4)$

□ **$SO(4)$ and its generators**:

$SO(4)$ rotation is expressed as

$$x'_\mu = \Lambda_{\mu\nu} x_\nu$$
$$\Lambda^T \Lambda = 1$$

Writing $\Lambda = \exp(\xi)$ and considering the infinitesimal transformation, we easily find that ξ is real 4×4 antisymmetric matrix. The standard basis for such antisymmetric matrices can be taken as $L_{\mu\nu}$ defined by (choosing a convenient overall sign)

$$(L_{\mu\nu})_{\rho\sigma} \equiv -(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) \tag{3.11}$$

In other words, **$L_{\mu\nu}$ has -1 at the position (μ, ν) and 1 at (ν, μ)** and 0 for all the other elements. The non-vanishing commutator for these generators is of the form

$$[L_{\mu\nu}, L_{\nu\rho}] = L_{\rho\mu} \qquad \text{no sum over } \nu \tag{3.12}$$

(For instance, $[L_{23}, L_{31}] = L_{12}$.) ξ can then be decomposed as (watch for the order of indices)

$$\xi = -\frac{1}{2}\xi_{\mu\nu}L_{\mu\nu} \tag{3.13}$$

$L_{\mu\nu}$ is a generator of rotation in the μ-ν plane. For example, Rotation in the 1-2 plane is

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\sim \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \theta L_{12} x \end{aligned} \tag{3.14}$$

□ ***SU*(2) × *SU*(2) decomposition**:

If we define I_i and K_i $(i = 1, 2, 3)$ as

$$\begin{aligned} I_i &\equiv \frac{1}{2}\epsilon_{ijk}L_{jk} \\ K_i &\equiv L_{0i} \end{aligned}$$

they satisfy the commutation relations

$$\begin{aligned} [I_i, I_j] &= \epsilon_{ijk}I_k \\ [I_i, K_j] &= \epsilon_{ijk}K_k \\ [K_i, K_j] &= \epsilon_{ijk}I_k \end{aligned}$$

Now define the following combinations

$$J_i^{\pm} \equiv \frac{1}{2}(I_i \pm K_i) \tag{3.15}$$

Then, we find that they generate separtely the **algebra** of $SU(2)$:

$$\begin{aligned} \left[J_i^{\pm}, J_j^{\pm}\right] &= \epsilon_{ijk}J_k^{\pm} \\ \left[J_i^{\pm}, J_j^{\mp}\right] &= 0 \end{aligned}$$

To see that they generate really the **group** $SU(2) \times SU(2)$, compute the exponent $-\frac{1}{2}\xi_{\mu\nu}L_{\mu\nu}$:

$$
\begin{aligned}
-\frac{1}{2}\xi_{\mu\nu}L_{\mu\nu} &= -\xi_{0i}K_i - \frac{1}{2}\xi_{ij}\epsilon_{ijk}I_k \\
&= \left(-\frac{1}{2}\xi_{ij}\epsilon_{ijk} - \xi_{0k}\right) J_k^+ + \left(-\frac{1}{2}\xi_{ij}\epsilon_{ijk} + \xi_{0k}\right) J_k^- \\
&\equiv \theta_k^+ J_k^+ + \theta_k^- J_k^-
\end{aligned}
$$

Since $\theta_k^\pm$ are **real and independent**, the decomposition is indeed $SU(2) \times SU(2)$. In this regard, recall that for the Lorentz group, they are complex conjugate of each other.

□ **Intertwiner**:

We will need a more explicit relation between $SO(4)$ and $SU(2) \times SU(2)$.
Let M_{AB} be an $SU(2)$ transformation matrix and u_B be the fundamental spinor representation:

$$
u'_A = M_{AB}u_B\,, \quad M^\dagger M = 1\,, \quad A, B = 1, 2 \qquad (3.16)
$$

We will use dotted indices such as $\dot{A}, \dot{B}$ for the second $SU(2)$.

The above decomposition means that there must exist a 4×4 **intertwining matrix** $\boldsymbol{T_{A\dot{B},\mu}}$ such that

$$
\begin{aligned}
TJ_i^+T^{-1} &\equiv \mathcal{J}_i^+ = \frac{\Sigma_i^+}{2i} \otimes \mathbf{1} \\
TJ_i^-T^{-1} &\equiv \mathcal{J}_i^- = \mathbf{1} \otimes \frac{\Sigma_i^-}{2i}
\end{aligned}
$$

or more explicitly

$$
\begin{aligned}
T_{A\dot{B},\mu}(J_i^+)_{\mu\nu} &= \left(\frac{\Sigma_i^+}{2i} \otimes \mathbf{1}\right)_{A\dot{B},C\dot{D}} T_{C\dot{D},\nu} \\
T_{A\dot{B},\mu}(J_i^-)_{\mu\nu} &= \left(\mathbf{1} \otimes \frac{\Sigma_i^-}{2i}\right)_{A\dot{B},C\dot{D}} T_{C\dot{D},\nu}
\end{aligned}
$$

where $\Sigma_i^\pm$ are 2×2 $SU(2)$ generators. A solution to these set of equations is, regarding $T_{A\dot{B},\mu}$ as four 2×2 matrices,

$$T_0 = 1\,, \quad T_i = -i\tau_i \tag{3.17}$$

$$\Sigma_i^+ = \tau_i\,, \quad \Sigma_i^- = -\tau_i^T \tag{3.18}$$

where τ_i are the Pauli matricies. Indeed $\Sigma_i^\pm$ satisfy the same algebra. Hereafter, we shall write

$$\boxed{\sigma_\mu \equiv T_\mu = (1, -i\tau_i) \qquad (3.19)}$$

In this way, we obtain the following explicit decomposition formula for general $SO(4)$ transformation:

$$x' = \Lambda x = e^{\theta_k^+ J_k^+} e^{\theta_k^- J_k^-} x \tag{3.20}$$

$$\begin{aligned} Tx' &= \sigma_\mu \Lambda_{\mu\nu} x_\nu = \left[T e^{\theta_k^+ J_k^+} T^{-1}\right]\left[T e^{\theta_k^- J_k^-} T^{-1}\right] Tx \\ &= \left[e^{\theta_k^+ \Sigma^+/2i} \otimes e^{\theta_k^- \Sigma^-/2i}\right] Tx \\ &= e^{\theta_k^+ \Sigma^+/2i} \sigma_\nu e^{\theta_k^- (\Sigma^-)^T/2i} x_\nu \end{aligned} \tag{3.21}$$

Removing x_ν, this can be written as

$$\sigma_\mu \Lambda_{\mu\nu} = M_+ \sigma_\nu M_-^T \tag{3.22}$$

$$M_\pm = SU(2)_\pm \tag{3.23}$$

Therefore, σ_μ transforms under bifundamental $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2)\times SU(2)$.

3.2.2 Quaternions

σ_μ defined above is deeply related to the **quaternions**, which forms an algebra (actually a field) denoted by **H**.

Quaternion $q \in \mathbf{H}$ is defined by

$$q = \sum_{\mu=0}^{3} q_\mu e_\mu = q_0 e_0 + \sum_{i=1}^{3} q_i e_i \tag{3.24}$$

where q_μ are **real** numbers and e_μ's satisfy the following closed algebra:

$$\boldsymbol{e_0 e_0 = e_0 \,, \quad e_i e_i = -e_0 \,,} \tag{3.25}$$

$$\boldsymbol{e_0 e_i = e_i e_0 = e_i \,, \quad e_i e_j = \epsilon_{ijk} e_k \ (i \neq j)} \tag{3.26}$$

Since $e_i e_j = -e_j e_i$, quaternion algebra is in general non-commutative. This multiplication rule can be summarized as (μ, ν on the RHS are not summed)

$$\boldsymbol{e_\mu e_\nu = -(-1)^{\delta_{\mu 0}} \delta_{\mu\nu} e_0 + \delta_{\mu 0}(1 - \delta_{\nu 0}) e_\nu + \delta_{\nu 0}(1 - \delta_{\mu 0}) e_\mu + \sum_\rho \epsilon_{0\mu\nu\rho} \mathbf{e}_\rho} \tag{3.27}$$

When one sets $e_2 = e_3 = 0$, then it becomes a complex number, with e_1 being the imaginary unit i. Hereafter summation over the repeated indices will be assumed, unless otherwise stated. It is clear that $\mathbf{H}$ is closed under multiplication.

Remark: One can easily check that if we ignore the fact that σ_μ has the index structure $\boldsymbol{(\sigma_\mu)_{A\dot{B}}}$, namely that row and column indices are acted on by different $SU(2)$ groups, σ_μ satisfies exactly the same algebra as quaternions defined above.

□ **Conjugation, (anti-)self-duality and norm**:

Quaternionic conjugate of q will be denoted by $q^\dagger$ and is defined by

$$\boldsymbol{q^\dagger \equiv q_\mu e_\mu^\dagger} \tag{3.28}$$

$$\boldsymbol{e_0^\dagger = e_0 \,, \qquad e_i^\dagger = -e_i} \tag{3.29}$$

Consider now the products $e_\mu e_\nu^\dagger$ and $e_\mu^\dagger e_\nu$. Due to the following group theoretical reasons, they have very simple interpretations:

If we go back to σ_μ interpretation of quaternions, $e_\mu e_\nu^\dagger$ is a quantity which transforms solely by $SU(2)_+$, like $M_+ e_\mu e_\nu^\dagger M_+^\dagger$. Thus, it is a singlet under $SU(2)_-$ and is therefore a mixture of $(0,0)$ and $(1,0)$:

$$(\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},\frac{1}{2}) = (1,1) \oplus (1,0) \oplus +(0,1) \oplus (0,0)$$

The former must be represented by $\delta_{\mu\nu}$ and the latter should be a **self-dual** antisymmetric tensor, which will be denoted by $2i\sigma_{\mu\nu}$.

Similarly, $e_\mu^\dagger e_\nu$ transforms under $SU(2)_-$ like $M_-^* e_\mu^\dagger e_\nu M_-^T$ and is a mixture of $(0,0)$ and $(0,1)$. The latter is the **anti-self-dual** part of $SO(4)$ tensor.

In fact, one easily verifies the following relations:

$$e_\mu e_\nu^\dagger = \delta_{\mu\nu} + 2i\sigma_{\mu\nu} \quad (3.30)$$

$$e_\mu^\dagger e_\nu = \delta_{\mu\nu} + 2i\bar{\sigma}_{\mu\nu} \quad (3.31)$$

$$\sigma_{\mu\nu} = \frac{1}{4i}(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) \quad (3.32)$$

$$\bar{\sigma}_{\mu\nu} = \frac{1}{4i}(e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu) \quad (3.33)$$

where $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are antisymmetric, hermitian and satisfy the following **duality properties**:

$$
\begin{aligned}
{}^*\boldsymbol{\sigma}_{\mu\nu} &\equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\boldsymbol{\sigma}_{\rho\sigma} = \boldsymbol{\sigma}_{\mu\nu} \quad \textbf{self-dual(SD)} \\
{}^*\bar{\boldsymbol{\sigma}}_{\mu\nu} &\equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\boldsymbol{\sigma}}_{\rho\sigma} = -\bar{\boldsymbol{\sigma}}_{\mu\nu} \quad \textbf{anti-self-dual(ASD)} \\
\textbf{where} \qquad \epsilon_{0123} &\equiv 1
\end{aligned}
$$

(For example, ${}^*(e_0e_1^\dagger) = e_2e_3^\dagger = -e_1 = e_0e_1^\dagger$ satisfying self-duality.)

't Hooft tensor: $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are traceless as 2×2 matrices. Thus, they can be expanded in terms of e_a, with $a = 1, 2, 3$. Explicitly,

$$\boldsymbol{\sigma}_{\mu\nu} \equiv \frac{1}{2}\eta^a_{\mu\nu}(ie_a) = \eta^a_{\mu\nu}\frac{\tau_a}{2} \tag{3.34}$$

$$\eta^a_{\mu\nu} = \delta_{\mu 0}\delta_{\nu a} - \delta_{\nu 0}\delta_{\mu a} + \epsilon_{0a\mu\nu} \tag{3.35}$$

$$\bar{\boldsymbol{\sigma}}_{\mu\nu} \equiv \frac{1}{2}\bar{\eta}^a_{\mu\nu}(ie_a) = \bar{\eta}^a_{\mu\nu}\frac{\tau_a}{2} \tag{3.36}$$

$$\bar{\eta}^a_{\mu\nu} = -(\delta_{\mu 0}\delta_{\nu a} - \delta_{\nu 0}\delta_{\mu a}) + \epsilon_{0a\mu\nu} \tag{3.37}$$

$\eta^a_{\mu\nu}, \bar{\eta}^a_{\mu\nu}$ are often called **'t Hooft tensors**. In the construction of instanton solution, e_a will be regarded as the basis for the **gauge group $SU(2)$**. Thus,'t Hooft tensors intertwine the spacetime and internal groups.

Properties of the 't Hooft tensors:

$$
\begin{aligned}
\eta_{a\mu\nu} &= \epsilon_{a\mu\nu}\,, \quad \text{if} \quad \mu,\nu = 1,2,3 \\
\eta_{a0\nu} &= \delta_{a\nu} \\
\eta_{a\mu 0} &= -\delta_{a\mu} \\
\eta_{a00} &= 0 \\
\bar{\eta}_{a\mu\nu} &= (-1)^{\delta_{\mu 0}+\delta_{\nu 0}}\eta_{a\mu\nu}
\end{aligned}
$$

$$\eta_{a\mu\nu}\eta_{b\mu\nu} = 4\delta_{ab}$$
$$\eta_{a\mu\nu}\eta_{a\mu\lambda} = 3\delta_{\nu\lambda}$$
$$\eta_{a\mu\nu}\eta_{a\mu\nu} = 12$$
$$\eta_{a\mu\nu}\eta_{a\kappa\lambda} = \delta_{\mu\kappa}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\kappa} + \epsilon_{\mu\nu\kappa\lambda}$$
$$\delta_{\kappa\lambda}\eta_{a\mu\nu} + \delta_{\kappa\nu}\eta_{a\lambda\mu} + \delta_{\kappa\mu}\eta_{a\nu\lambda} + \eta_{a\sigma\kappa}\epsilon_{\lambda\mu\nu\sigma} = 0$$
$$\eta_{a\mu\nu}\eta_{b\mu\lambda} = \delta_{ab}\delta_{\nu\lambda} + \epsilon_{abc}\eta_{c\nu\lambda}$$
$$\epsilon_{abc}\eta_{b\mu\nu}\eta_{c\kappa\lambda} = \delta_{\mu\kappa}\eta_{a\nu\lambda} - \delta_{\mu\lambda}\eta_{a\nu\kappa} - \delta_{\nu\kappa}\eta_{a\mu\lambda} + \delta_{\nu\lambda}\eta_{a\mu\kappa}$$

$$\begin{aligned} \eta_{a\mu\nu}\bar{\eta}_{b\mu\nu} &= 0 \\ \eta_{a\kappa\mu}\bar{\eta}_{b\kappa\lambda} &= \eta_{a\kappa\lambda}\bar{\eta}_{b\kappa\mu} \end{aligned}$$

Norm: The norm-squared of q is defined as $q^\dagger q$:

$$|q|^2 = q^\dagger q = q_\mu q_\nu e_\mu^\dagger e_\nu = q_\mu q_\nu \delta_{\mu\nu} = \sum q_\mu q_\mu \geq 0 \quad (3.38)$$

where the equality holds if and only if $q = 0$. It is clear that $q^\dagger q = qq^\dagger$ holds. Furthermore, this shows that non-vanishing q always has the inverse q^{-1} (hence $\mathbf{H}$ forms a field) given by

$$q^{-1} = \frac{q^\dagger}{|q|^2}, \qquad qq^{-1} = q^{-1}q = 1 \quad (3.39)$$

□ **Another view of 't Hooft tensor**:

The 't Hooft tensor can be introduced from a slightly different point of view. The idea is to extend the action of $SO(4)$ to the internal gauge group part. One natural way is to **intertwine the gauge group $SU(2)_g$ with one of the $SU(2)$ factor of $SO(4)$**.

For instance, extend $SU(2)_+$ to $SU(2)_+ \oplus SU(2)_g$. The total generator for this sector becomes

$$\tilde{J}_i^+ = J_i^+ + t_i \quad (3.40)$$

where t_i is the generator of $SU(2)_g$ satisfying $[t_i, t_j] = \epsilon_{ijk}t_k$. $SU(2)_-$ sector is unchanged, *i.e.* $\tilde{J}_i^- = J_i^-$. Then, going backwards to $L_{\mu\nu}$, we easily find the following modified expressions denoted by $\tilde{L}_{\mu\nu}$:

$$\begin{aligned} \tilde{L}_{\mu\nu} &= L_{\mu\nu} + l_{\mu\nu} \\ \text{where} \qquad l_{ij} &= \epsilon_{ijk}t_k \\ l_{0i} &= t_i \end{aligned}$$

Since $\tilde{J}_i^{\pm}$ satisfy exactly the same commutation relations as before, $\tilde{L}_{\mu\nu}$ (and hence $l_{\mu\nu}$ themselves) obey $SO(4)$ algebra. Now introduce $\eta_{a\mu\nu}$ by

$$l_{\mu\nu} = \eta_{a\mu\nu}t_a = \eta_{a\mu\nu}\frac{\tau_a}{2i} \tag{3.41}$$

Then, one can check that $\eta_{a\mu\nu}$ is exactly the 't Hooft tensor. This also means the identification $\boldsymbol{\sigma_{\mu\nu} = il_{\mu\nu}}$.

□ **Conversion between e_μ and $e_\mu^\dagger$:**

The following relation is often useful:

$$\begin{aligned} \epsilon e_\mu \epsilon^T &= e_\mu^*, \qquad \epsilon = i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \Leftrightarrow \quad \epsilon_{AA'}(e_\mu)_{A'B'}\epsilon_{B'B} &= -(e_\mu^\dagger)_{BA} \end{aligned}$$

3.3 (Anti-)Self-Dual Configurations as Classical Solutions

3.3.1 Some formulas

We define the dual field strength as

$$\tilde{\boldsymbol{F}}_{\mu\nu} \equiv \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}} \tag{3.42}$$

Then we find

$$\boldsymbol{F^2_{\mu\nu} \;=\; \tilde{F}^2_{\mu\nu}} \tag{3.43}$$

Proof:

$$\tilde{F}^2_{\mu\nu} \;=\; \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} = \frac{1}{4}2(\delta_{\alpha\rho}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\rho})F_{\alpha\beta}F_{\rho\sigma} = F^2_{\mu\nu}$$

Using this formula, we get (with $F = F^a_{\mu\nu}$ etc.)

$$\frac{1}{2}(F \pm \tilde{F})^2 \;=\; \frac{1}{2}(F^2 + \tilde{F}^2 \pm 2F\tilde{F}) = F^2 \pm F\tilde{F} \geq 0 \tag{3.44}$$

Since $F^2 \geq 0$, this implies

$$\boldsymbol{F^2 \geq |F\tilde{F}|} \tag{3.45}$$

where the equality holds when $F \pm \tilde{F} = 0$, *i.e.* for $\boldsymbol{F^a_{\mu\nu} = \pm\tilde{F}^a_{\mu\nu}}$. These are called **self-dual (SD)** and **anti-self-dual (ASD)** configurations. Hereafter, we use (A)SD to denote both of these configurations.

3.3.2 Minimum action configurations

Integrate the relation (3.44) above over the space-time. We get

$$S_E \;=\; \frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu} \geq |Q| \tag{3.46}$$

$$\text{where}\qquad Q \;\equiv\; \frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} \tag{3.47}$$

and the minimum value of the action is attained for (A)SD configurations.

□ **Equation of motion and Bianchi identity**:

(A)SD configurations are necessarily **solutions** of the classical YM equation

$$[D_\mu\,, F_{\mu\nu}] \;=\; 0 \tag{3.48}$$

This is because the Bianchi identity

$$0 = \left[D_\mu \, , \tilde{F}_{\mu\nu}\right] = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\left[D_\mu \, , [D_\alpha \, , D_\beta]\right] \qquad (3.49)$$

is equivalent to the equation of motion for (A)SD configurations.

□ **Vanishing of the energy-momentum tensor**:

The energy-momentum tensor is given by (omitting the group theory superscript a)

$$T_{\mu\nu} = F_{\mu\lambda}F_{\lambda\nu} - \frac{1}{4}\delta_{\mu\nu}(F_{\alpha\beta}F_{\beta\alpha}) \qquad (3.50)$$

Exercise: Prove that $T_{\mu\nu} = 0$ for (A)SD configurations.

Now we use the following identity:

$$\tilde{F}_{\mu\lambda}\tilde{F}_{\lambda\nu} = \frac{1}{2}\delta_{\mu\nu}F_{\alpha\beta}F_{\beta\alpha} - F_{\mu\alpha}F_{\alpha\nu} \qquad (3.51)$$

This can be proved by direct calculation:

$$\begin{aligned} \tilde{F}_{\mu\lambda}\tilde{F}_{\lambda\nu} &= \frac{1}{4}\epsilon_{\mu\lambda\alpha_1\alpha_2}\epsilon_{\lambda\nu\beta_1\beta_2}F_{\alpha_1\alpha_2}F_{\beta_1\beta_2} \\ &= -\frac{1}{4}\begin{vmatrix} \delta_{\mu\nu} & \delta_{\mu\beta_1} & \delta_{\mu\beta_2} \\ \delta_{\alpha_1\nu} & \delta_{\alpha_1\beta_1} & \delta_{\alpha_1\beta_2} \\ \delta_{\alpha_2\nu} & \delta_{\alpha_2\beta_1} & \delta_{\alpha_2\beta_2} \end{vmatrix} F_{\alpha_1\alpha_2}F_{\beta_1\beta_2} \\ &= \cdots \\ &= \frac{1}{2}\delta_{\mu\nu}F_{\alpha\beta}F_{\beta\alpha} - F_{\mu\alpha}F_{\alpha\nu} \end{aligned} \qquad (3.52)$$

Therefore, we get

$$\frac{1}{4}\delta_{\mu\nu}(F_{\alpha\beta}F_{\beta\alpha}) = \frac{1}{2}(F_{\mu\alpha}F_{\alpha\nu} + \tilde{F}_{\mu\alpha}\tilde{F}_{\alpha\nu}) \qquad (3.53)$$

Putting this into $T_{\mu\nu}$, we find

$$T_{\mu\nu} = \frac{1}{2}(F_{\mu\alpha}F_{\alpha\nu} - \tilde{F}_{\mu\alpha}\tilde{F}_{\alpha\nu}) \qquad (3.54)$$

which obviously vanishes for (A)SD configurations.

3.4 Winding Number for Finite Action Configurations

3.4.1 Topological nature of the charge Q

It is easy to see that Q is topological in the sense that it is invariant under any continuous deformation of A_μ. Infact

$$\begin{aligned}
\delta \mathrm{Tr} F_{\mu\nu}\tilde{F}_{\mu\nu} &= 2\mathrm{Tr}(\delta\left[D_\mu, D_\nu\right]\tilde{F}_{\mu\nu}) \\
&= 4\mathrm{Tr}(\left[\partial_\mu + A_\mu, \delta A_\nu\right]\tilde{F}_{\mu\nu}) \\
&= \partial_\mu \mathrm{Tr}(4\delta A_\nu \tilde{F}_{\mu\nu}) + 4\mathrm{Tr}(\delta A_\nu \left[D_\mu, \tilde{F}_{\mu\nu}\right]) \qquad (3.55)
\end{aligned}$$

Due to the Bianchi identity, the second term vanishes and the result is a total derivative. Upon integration this vanishes if at least $F_{\mu\nu}$ tends to zero at infinity. Note that this is true for any configuration.

Remark: We have already discussed this in the lecture on anomaly. There we derived the formula

$$\delta \mathrm{Tr}(F^{n+1}) = d\left((n+1)\mathrm{Tr}(\delta A F^n)\right) \qquad (3.56)$$

For $n = 1$, this is nothing but the above equation:

$$\begin{aligned}
F^2 &= F \wedge F = \frac{1}{2} F_{\mu\nu}\tilde{F}_{\mu\nu} d^4x \\
\delta A F &= \delta A \wedge F = \frac{1}{2} A_\mu F_{\alpha\beta} dx^\mu dx^\alpha dx^\beta \\
\therefore \qquad d(\delta A F) &= \partial_\nu(\delta A_\mu \tilde{F}_{\nu\mu}) d^4x
\end{aligned}$$

In fact, we showed that $\mathrm{Tr}F^2 = d\omega_3^0$, where

$$\omega_3^0 = \mathrm{Tr}\left(AF - \frac{1}{3}A^3\right) = \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) = \text{Chern-Simons form}$$

More explicitly,

$$\begin{aligned} d\omega_3^0 &= \partial_\mu K_\mu d^4x \\ K_\mu &= \epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(\frac{1}{2}A_\nu F_{\alpha\beta} - \frac{1}{3}A_\nu A_\alpha A_\beta\right) \end{aligned}$$

3.4.2 Non-trivial gauge transformations and their winding number

For the action to be finite, $F_{\mu\nu}$ must fall off faster than $1/r^2$ as $r = \sqrt{x^2} \to \infty$. This means that **the gauge potential must fall off faster than $1/r$ up to a gauge transformation**, *i.e.*

$$\boldsymbol{A_\mu \sim g^{-1}\partial_\mu g + o\left(\frac{1}{r}\right)}$$

□ **Case of $G = SU(2)$**:

The most general $SU(2)$ gauge transformation can be written as

$$g = a + ib_i\tau_i$$

where a, b_i $(i = 1, 2, 3)$ are real numbers satisfying

$$a^2 + b_ib_i = 1 \quad \Leftarrow \quad g^\dagger g = 1$$

This shows that $SU(2)$ is topologically a 3-sphere S^3.

Note that **g is nothing but a quaternion q with the unit norm $q^\dagger q = 1$**. This is the well-known equivalence

$$SU(2) \simeq Sp(1)$$

When a and b_i become functions of x_μ, $g(x)$ for large fixed r gives a mapping **S^3(space time) $\longrightarrow$ $S^3(SU(2))$**. The important fact is that such gauge transformations are classified by the

homotopy group, *i.e.* the additive group of equivalence class whose members are continuously deformable to each other.

The simplest non-trivial gauge transformation which is not homotipic to a constant can be represented by

$$\boldsymbol{g = \frac{1}{r}(x_0 + i\vec{x}\cdot\vec{\tau}) = \frac{1}{r}\sigma^\dagger_\nu x_\nu}$$

It is clear that, for a fixed r, as one covers S^3 in space one covers $SU(2)$ group space exactly once.

Let us compute the **pure gauge potential corresponding to this** $\boldsymbol{g}$:

$$\begin{aligned}
g^{-1} &= \frac{1}{r}\sigma_\nu x_\nu \\
\partial_\mu g &= \frac{1}{r}\sigma^\dagger_\mu - \frac{x_\mu}{r^3}\sigma^\dagger_\nu x_\nu \\
A_\mu &= g^{-1}\partial_\mu g = \frac{1}{r}\sigma_\lambda x_\lambda\left(\frac{1}{r}\sigma^\dagger_\mu - \frac{x_\mu}{r^3}\sigma^\dagger_\nu x_\nu\right) = \cdots \\
&= \frac{-2i\sigma_{\mu\lambda}x_\lambda}{r^2} = \frac{2l_{\mu\lambda}x_\lambda}{r^2}
\end{aligned}$$

where we used $\sigma_\mu\sigma^\dagger_\nu = \delta_{\mu\nu} + 2i\sigma_{\mu\nu}$.

Since, as we have seen, the topological charge is a homotopy invariant, it must characterize the **homotopy class** of the gauge transformation g. Since except at the origin F vanishes for a pure gauge potential, we have

$$\begin{aligned}
\int \mathrm{Tr}F^2 &= \int d^4x\partial_\mu K^g_\mu = \int_{S^3} d^3x\frac{x_\mu}{r}K^g_\mu \\
K^g_\mu &= -\frac{1}{3}\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}(A_\nu A_\alpha A_\beta)
\end{aligned}$$

(AF part of K_μ is zero for pure gauge configuration.)
To evaluate this, note that
(i) K^g_μ is a vector and hence $x_\mu K^g_\mu$ is rotationally invariant and
(ii) we may set $r = 1$ since we are computing a homotopy invariant.
Thus, all we have to do is to compute the value of the integrand at one point on a unit S^3 and multiply by the volume of S^3, which is $2\pi^2$. Take the point to be $x_0 = 1, x_i = 0$. Then,

$$\begin{aligned} A_i &= -2i\sigma_{i0} = i\tau_i \\ \frac{x_\mu}{r} K^g_\mu &= K^g_0 = \frac{i}{3}\epsilon_{ijk}\mathrm{Tr}(\tau_i\tau_j\tau_k) = -\frac{2}{3}\epsilon_{ijk}\epsilon_{ijk} = -4 \end{aligned}$$

Thus, if we define the **Pontryagin index (or winding number) k** by

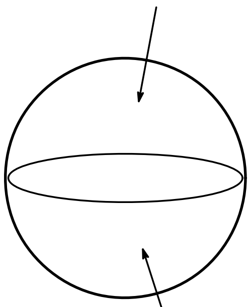
$$\boldsymbol{k \equiv \frac{1}{32\pi^2}\int d^4x\, F^a_\mu \tilde{F}^a_{\mu\nu} = -\frac{1}{8\pi^2}\int \mathrm{Tr}F^2}$$

we get

$$k = -\frac{1}{8\pi^2}(-4)2\pi^2 = 1$$

for the above homotopy class.

Additivity of the winding number: For more general gauge transformation, the following observation suffices: Let the winding number of $g_i, i = 1, 2$ be k_i and consider the product $g = g_1 g_2$.



Since the winding number is unchanged by continuous deformation, we may deform $g_1(g_2)$ such that $g_1 = 1(g_2 = 1)$on the lower (upper) hemisphere of S^3. In this case the winding number k_1 for g_1 is obtained by integration over the upper hemisphere only and so on. It is then clear that the winding number of $g_1 g_2$ is $k_1 + k_2$.

3.5 One Instanton Solution for $SU(2)$

With these preparations, we now describe how to obtain the simplest (anti-)instanton solution with $k = \pm 1$.

Since the (A)SD equations are still rather difficult to solve in complete generality, one would like to make an **ansatz** to find solutions.

The most natural strategy is to first look for a self-dual solution with $SO(4)$ symmetry. An obvious ansatz (adopted by BPST) to try is

$$\boldsymbol{A_\mu = gA^a_\mu t_a = \eta^a_{\mu\nu} t_a x_\nu f(x^2) = l_{\mu\nu} x_\nu f(x^2)}$$

which satisfies the gauge condition $x_\mu A_\mu = 0$. Using the $SO(4)$ commutation relations, we can easily compute $F_{\mu\nu}$ to be

$$F_{\mu\nu} = \underbrace{l_{\mu\nu}}_{SD}(x^2 f^2 - 2f) + \underbrace{(x_\mu l_{\nu\lambda} x_\lambda - (\mu \leftrightarrow \nu))}_{ASD}(2f' + f^2)$$

where f' means derivative with respect to x^2. For this to be self-dual, we must set $2f' + f^2 = 0$. The general solution of this equation is

$$f(x^2) = \frac{2}{x^2 + \rho^2} \tag{3.57}$$

with ρ a constant. Thus we get a regular self-dual solution

$$\boldsymbol{A_\mu = \frac{2 l_{\mu\nu} x_\nu}{x^2 + \rho^2}, \quad A^a_\mu = \frac{2}{g} \frac{\eta^a_{\mu\nu} x_\nu}{x^2 + \rho^2}} \tag{3.58}$$

$$\boldsymbol{F_{\mu\nu} = -\frac{4 l_{\mu\nu} \rho^2}{(x^2 + \rho^2)^2}, \quad F^a_{\mu\nu} = -\frac{4}{g} \frac{\eta^a_{\mu\nu} \rho^2}{(x^2 + \rho^2)^2}} \tag{3.59}$$

- $\boldsymbol{\rho}$ can be interpreted as the **size** of the instanton.
- From translation invariance, we may replace x_μ by $x_\mu - a_\mu$ with $\boldsymbol{a_\mu}$ describing the **position** of the instanton.
- Thus, this solution has **5 gauge-invariant free parameters**, called the **moduli** of an instanton solution.
- With $\bar{\eta}^a_{\mu\nu}$ replacing $\eta^a_{\mu\nu}$, one gets the anti-instanton solution.

Note that **as $r \to \infty$, A_μ precisely reduces to the pure gauge $g^{-1}\partial_\mu g$ carrying winding number 1, with g discussed previously**.

3.6 A Class of Multi-Instanton Solutions

3.6.1 Extended Ansatz

A more general ansatz which yields a class of multi-instanton solutions is of the form[1]

$$\boldsymbol{A_\mu \;=\; l_{\mu\alpha}\partial_\alpha f(x)} \tag{3.60}$$

which satisfies the gauge condition $\partial_\mu A_\mu = 0$. One can easily compute $F_{\mu\nu}$ to be

$$\begin{aligned} F_{\mu\nu} &= l_{\mu\nu}(\partial f)^2 - l_{\mu\rho}S_{\rho\nu} + l_{\nu\rho}S_{\rho\mu} \\ \text{where} \qquad S_{\mu\nu} &= \partial_\nu\partial_\nu f + \partial_\mu f \partial_\nu f = \text{symmetric} \end{aligned}$$

Now decompose $V_{\alpha\beta}$ into the traceless part and the trace part:

$$\begin{aligned} S_{\mu\nu} &= T_{\mu\nu} + \frac{1}{4}\delta_{\mu\nu}S \\ S &= S_{\mu\mu}\,, \qquad T_{\mu\mu} = 0 \end{aligned}$$

Then, we get

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{2}l_{\mu\nu}((\partial f)^2 - \partial^2 f) + A_{\mu\nu} \\ A_{\mu\nu} &= -l_{\mu\rho}T_{\rho\nu} + l_{\nu\rho}T_{\rho\mu} \end{aligned}$$

The first term is clearly self-dual. Although it is not at all obvious, **$A_{\mu\nu}$ part is actually anti-self-dual**. This can be checked by studying $\tilde{A}_{i0}$ and $\tilde{A}_{ij}$ separately. The reason for it is

[1] F. Wilczeck, in "Quark confinement and Field Theory", ed. D. Stump and D. Weingarten (New York, 1977); E. Corrigan and D.B. Fairlie, PLB67 (77) 69.

roughly as follows: In terms of $SU(2) \times SU(2)$ representations, $l_{\mu\nu} \in (1,0)$ and $T_{\mu\nu}$(traceless, symmetric) $\in (1,1)$. Thus $(1,0) \times (1,1) = (0,1) \oplus (2,1)$. The above combination picks up the anti-self-dual part $(0,1)$.

3.6.2 Self-dual solution

Self-dual solution is obtained if we set $A_{\mu\nu} = 0$, *i.e.* $T_{\mu\nu} = 0$:

$$S_{\mu\nu} - \frac{1}{4}\delta_{\mu\nu}S = \partial_\mu\partial_\nu f + \partial_\mu f \partial_\nu f - \frac{1}{4}\delta_{\mu\nu}\left(\partial^2 f + (\partial f)^2\right) = 0$$

It is convenient to set $\boldsymbol{f = -\ln\varphi}$. Then, the equation above can be rewritten as

$$\partial_\mu\left(\frac{\partial_\nu\varphi}{\varphi^2}\right) = \frac{1}{4}\delta_{\mu\nu}\partial_\rho\left(\frac{\partial_\rho\varphi}{\varphi^2}\right)$$

This means that $\partial_\nu\varphi/\varphi^2$ can only be a linear function of x_ν of the form $cx_\nu + d_\nu$. Thus, we have

$$\frac{\partial_\nu\varphi}{\varphi^2} = \partial_\nu(-\varphi^{-1}) = cx_\nu + d_\nu$$

For $c \neq 0$, the solution is of the form

$$\varphi = -\frac{1}{\frac{1}{2}c(x-a)^2 + b}$$

This gives a finite action only if the sign of c and b are the same, so that φ never blows up. In such a case, it coincides with the BPST solution, which is not new. For $c = 0$, $F_{\mu\nu}$ becomes singular.

3.6.3 Anti-self-dual solution

Another possiblity is to set the self-dual part to zero. This will turn out to give more interesting solutions. The equation is $(\partial f)^2 = \partial^2 f$ and just as before, set $f = -\ln\varphi$. Then this simplifies to

$$\frac{\partial^2\varphi}{\varphi} = 0$$

The most general solution with positive definite sign with isolated singularities is

$$\varphi = \sum_{i=1}^{N} \frac{\rho_i^2}{(x-a_i)^2} + c^2$$

- Due to the division by φ, the δ-function is annihilated and this is a legitimate solution of the above equation.

- Since the equation above is defined only up to an overall constant, there are actually only **two types of solutions**: $c = 1$ (first considered by 't Hooft) and $c = 0$ (introduced by Jackiw, Nohl and Rebbi).

As we shall see, **they represent multi-anti-instanton solutions with winding number $-N$ and $-(N-1)$** respectively.

3.6.4 Regular Solution by Gauge Transformations

The ASD solution above is singular at N points $x = a_i$. Actually, these singularities are **gauge artifacts**.

To show this, we must be rather careful and define the gauge field B_μ which is equal to A_μ **except at the singular points**. Explicitly,

$$B_\mu = -\frac{2i\sigma_{\mu\nu}}{\varphi}\sum_{i=1}^{N}\frac{\rho_i^2(x-a_i)_\nu}{(x-a_i)^4}, \qquad x \neq a_i$$

(This form is exactly what we get if we formally compute A_μ.)

***c* = 0 case:** Consider first the $c = 0$ case. Then, it has the following asymptotic behavior as $x \to \infty$:

$$\varphi \xrightarrow{x\to\infty} \frac{1}{x^2}\sum_{i=1}^{N}\rho_i^2$$
$$B_\mu \xrightarrow{x\to\infty} -\frac{2i\sigma_{\mu\nu}x_\nu}{x^2}$$

This shows, surprisingly at first, that B_μ has the same asymptotic behavior as the BPST instanton (not anti-instanton) despite the fact that we are dealing with ASD solution[2]. In any case, this means that B_μ approaches a pure gauge

$$B_\mu \xrightarrow{x\to\infty} = g^{-1}\partial_\mu g$$
$$g = \frac{x_0 + i\tau_i x_i}{r}$$

Now define $\bar{k}$ to be the winding number **as defined solely by the asymptotic behavior**. Then, obviously,

$$\bar{k}(B_\mu) = 1$$

***c* = 1 case:** The case of $c = 1$ is more puzzling. In this case,

$$\varphi \xrightarrow{x\to\infty} 1 + \mathcal{O}(1/x^2) \quad \Rightarrow \quad B_\mu \xrightarrow{x\to\infty} \mathcal{O}(1/x^3)$$

[2] $F_{\mu\nu}$ is indeed still ASD.

and hence $\bar{k} = 0$.

What is happening is that **$\bar{k}$ need not coincide with the true winding number k which is properly defined only for a regular solution.** To see this, we must study whether we can remove the singularity by a gauge transformation. We will do this one at a time.

First look at **the behavior around $x = a_1$**. One easily finds (for general c)

$$\varphi \quad \xrightarrow{x \to a_1} \quad \frac{\rho_1^2}{(x-a_1)^2} + \underbrace{\sum_{j=2}^{N} \frac{\rho_j^2}{(a_1-a_j)^2} + c^2}_{c_1^2} + \mathcal{O}\left(\frac{|x-a_1|}{a}\right)$$

$$\text{where} \qquad a \quad \equiv \quad \min_{j \neq 1} |a_1 - a_j|$$

For simplicity, let us define

$$\begin{aligned} c_1^2 &\equiv c^2 + \sum_{j=2}^{N} \frac{\rho_j^2}{(a_1-a_j)^2} \\ \rho^2 &\equiv \frac{\rho_1^2}{c_1^2} \\ y &\equiv x - a_1 \end{aligned}$$

Then the behavior above takes the form

$$\varphi \quad \xrightarrow{y \to 0} \quad c_1^2 \left(1 + \frac{\rho^2}{y^2}\right) + \mathcal{O}(|y|/a)$$

From this one finds

$$\begin{aligned} B_\mu \quad \xrightarrow{y \to 0} \quad & \frac{-2i\rho^2 \sigma_{\mu\nu} y_\nu}{y^2(y^2+\rho^2)} + \mathcal{O}(y^2/a^2) \\ = \quad & \frac{-2i\sigma_{\mu\nu} y_\nu}{y^2} + \frac{y^2}{y^2+\rho^2}\left(\frac{-2i\sigma_{\mu\nu} y_\nu}{y^2}\right) + \mathcal{O}(y^2/a^2) \end{aligned}$$

This means that the singular part of B_μ at $y = 0$ is a pure gauge and can be removed by the **inverse** of the "large" gauge transformation $g(y)$ given previously. (This procedure should be regarded as a mere technique of getting a regular solution.) The result of this procedure is

$$\begin{aligned} B'_\mu &= gB_\mu g^{-1} + g\partial_\mu g^{-1} \\ &= \frac{y^2}{y^2+\rho^2} g\partial_\mu g^{-1} + \mathcal{O}(y^2/a^2) \\ &= \frac{2i\bar{\sigma}_{\mu\nu}y_\nu}{y^2+\rho^2} + \mathcal{O}(y^2/a^2) \qquad \Leftarrow g\partial_\mu g^{-1} = \frac{2i\bar{\sigma}_{\mu\nu}y_\nu}{y^2} \end{aligned}$$

Thus by this procedure, we indeed get around $y = 0$ a **regular anti-self-dual structure**. Now since we have performed a gauge transformation by g^{-1}, the winding number $\bar{k}$ is now decreased by one unit. Thus **every time we remove the singularity by a gauge transformation we have $\Delta\bar{k} = -1$**. So, after removing N singularities, we get

$$\begin{aligned} k &= \bar{k} = 1 - N = -(N-1) \qquad \text{for } c = 0 \\ k &= \bar{k} = 0 - N = -N \qquad\qquad \text{for } c = 1 \end{aligned}$$

Exercise: Compute the winding number directly from the singular solution. (Hint: Utilize the fact that $\partial^2\partial^2 \ln \prod_{i=1}^N (x - a_i)^2 = 0$ for $x \neq a_i$.)